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A finite decentralized marriage market with bilateral search

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Abstract

I study a model in which a finite number of men and women look for future spouses via random pairwise meetings. The central question is whether equilibrium marriage outcomes are stable matchings when search frictions are small. The answer is they can but need not be. For any stable matching there is an equilibrium leading to it almost surely. However there may also be equilibria leading to an unstable matching almost surely. A restriction to simpler strategies or to markets with aligned preferences rules out such equilibria. However unstable—even Pareto-dominated—matchings may still arise with positive probability under those two restrictions combined. In addition, inefficiency due to delay may remain significant despite vanishing search frictions. Finally, a condition is identified under which all equilibria are outcome equivalent, stable, and efficient.

JEL classification: C78; D83

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1. Introduction

1.1. Overview

The stable matching is the main solution concept for cooperative two-sided matching problems under nontransferable utility. Many centralized mechanisms are designed to implement

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stable matchings.[1] However, whether outcomes of decentralized two-sided matching markets correspond to stable matchings remains unclear. The present paper addresses this question by considering a decentralized two-sided matching market modeled as a search and matching game. Following Gale and Shapley (1962) I inherit the interpretation that the game represents the situation in which unmarried men and women gather in a marketplace to look for future spouses. The game starts with an initial market à la Gale–Shapley, henceforth referred to as a marriage market, consisting of finitely many men and women with heterogeneous preferences. In every period a meeting between a randomly selected pair of a man and a woman takes place, during which they sequentially decide whether to marry each other. Mutual agreement leads to marriage. Married couples leave the game. Disagreement leads to separation. Separated people continue searching. The game ends when no mutually acceptable pairs of a man and a woman are left. Search is costly due to frictions parametrized as a common discount factor that diminishes the value of a future marriage. A game outcome, reflecting who has married whom and who stays single, corresponds to a matching for the initial market. The central question addressed in the paper is whether matchings that obtain in equilibria are stable matchings for the initial market when search frictions are small. The analysis focuses on a near-frictionless setting in order to test the general conjecture that if in a decentralized market the participants have easy access to each other with low costs then equilibrium outcomes would be in the core of the underlying market.[2] The paper shows that the answer to the central question is indeterminate at best and No in general, in contrast to what has been conjectured on this matter.[3] First, for any stable matching there is an equilibrium leading to that matching almost surely (Proposition 4.2), that is, every player expects to marry according to the pairing scheme implied by the matching. This result establishes that the set of all stable matchings is contained in the set of all matchings that may arise in equilibria. Then it is shown that the latter set may contain unstable matchings as well: Under certain preference structures there are equilibria leading to an unstable matching almost surely (Example 1). The paper proceeds to propose two conditions, each of which rules out such equilibria: 1. The players do not condition their behavior on the actions during any past failed meeting (Proposition 4.4). 2. The players' preferences satisfy the Sequential Preference Condition, a condition that implies a certain degree of preference alignment (Proposition 4.6). However the two conditions, separately or combined, are not sufficient to rule out equilibria in which unstable matchings arise with positive probability; some of the probable matchings may even be Pareto-dominated (Example 3). Another source of inefficiency is delay: Significant loss of efficiency due to delay may be present in an equilibrium even if search frictions are arbitrarily small (Examples 4 and 5). The paper ends with a uniqueness result that is pro-stability and efficiency: If the players' preferences satisfy a strengthening of the Sequential Preference Condition which implies a stronger degree of alignment, then all equilibria are outcome equivalent, stable, and efficient (Proposition 4.8).

1.2. Literature

The present paper contributes to the literature on search and matching games in which a marriage market is embedded. The central question of the literature agrees with that of the present paper: Do equilibrium outcomes correspond to stable matchings? An early paper in this lit-

[1] See Roth and Liveria Sotomayor (1990) for a survey of the theory of the stable matching. Roth (2008) surveys applications in designing centralized mechanisms for two-sided matching markets.

[2] It is well known that the core of the marriage market is the set of all stable matchings.

[3] Roth and Liveria Sotomayor (1990, page 245).

erature (Roth and Vande Vate, 1990) studies the steady state of a search and matching game with short-sighted players and concludes that a stable matching obtains almost surely. Later papers consider sophisticated players. McNamara and Collins (1990), Burdett and Coles (1997), Eeckhout (1999), Bloch and Ryder (2000) and Smith (2006) assume that the underlying marriage market admits a unique stable matching that is positively assortative. Their results confirm that equilibrium outcomes retain some extent of assorting. Adachi (2003) and Lauermann and Nöldeke (2014) consider a market with a general preference structure. Adachi (2003) studies a model in which the steady state stock of active players is exogenously maintained and confirms that equilibrium outcomes converge to stable matchings as search frictions vanish. Lauermann and Nöldeke (2014) consider endogenous steady states and finds that all limit outcomes are stable if and only if the underlying market has a unique stable matching.

The model considered in this paper also embeds a marriage market in a search and matching game. In contrast to the previously cited papers, all of which study steady state equilibria in a stationary setting, the present model features a nonstationary search situation. Indeed, the market shrinks as players marry and leave. Moreover, all but Roth and Vande Vate (1990) consider a market with a continuum of nameless players, whereas in the present model the market is finite and the players are identifiable. Nonstationarity and finiteness make the present model qualitatively different from most models considered in the literature. It follows that the set of matchings that may obtain in equilibria of the present model is in general different from that of a stationary and continuum model.

Another related literature investigates models embedding a marriage market in a sequential bargaining game reminiscent of the deferred acceptance protocol in Gale and Shapley (1962). This literature includes Alcalde (1996), Diamantoudi et al. (2015), Pais (2008), Suh and Wen (2008), Niederle and Yariv (2009), Bloch and Diamantoudi (2011), and Haeringer and Wooders (2011).[4] Like the present paper, these papers consider a finite marriage market that shrinks as players marry and leave. The difference between models in this literature and those in the search and matching literature, including the present model, is the search technology. A sequential bargaining game models a market with directed search: When it is his or her turn to move, a player can reach and deal with any player of the opposite sex without delay or uncertainty. In contrast, a search and matching game models a market with undirected search: Bilateral meetings are stochastic; one needs patience and luck to encounter a particular person. One common finding among papers with a sequential bargaining model is that some or all stable matchings can be supported in equilibria. Such equilibria bear resemblance to equilibria that lead to a particular stable matching almost surely in the present model, see Proposition 4.2. On the other hand, unstable matchings may also obtain in equilibria of a sequential bargaining game, which is the case in Diamantoudi et al. (2015), Suh and Wen (2008) and Haeringer and Wooders (2011). This common finding is also in accordance with results in the present paper. However, because of random search, the model in the present paper may have equilibria that have no counterpart in a sequential bargaining model. For instance, in a typical sequential bargaining model, an equilibrium in pure strategies leads to one matching deterministically, whereas in this model an equilibrium in pure strategies may lead to several possible matchings, because the players' strategies may depend on which of the multiple probable paths the history has taken. In this respect, nonsta-

[4] Marriage problems belong to a class of coalitional games under nontransferable utility called "hedonic games". Alcalde and Romero-Medina (2000) and Alcalde and Romero-Medina (2005) investigate decentralized implementation of stable outcomes of many-to-one matching problems, which are also hedonic games. Bloch and Diamantoudi (2011) study implementability of the core of a general hedonic game.

tionarity has little influence in a sequential bargaining model because a player's expected payoff remains unchanged as the game unfolds, whereas in the present model exogenous uncertainty may drastically change a player's continuation prospect.

A third related literature[5] studies whether the Walrasian price can be supported in equilibria of a search and bargaining game in which an exchange economy, instead of a marriage market, is embedded. Papers from this literature and the present paper are united under the theory of non-cooperative foundation of cooperative solution concepts. Indeed, the present model can be seen as the nontransferable utility version of the models considered in Rubinstein and Wolinsky (1990) and Gale and Sabourian (2006).

The layout of the paper is as follows: Section 2 introduces the game. Section 3 sets up an analytic framework. Section 4 provides the analysis. Section 5 concludes. Lengthy proofs and additional examples are found in Appendix A.

2. The game

2.1. The marriage market

There are two disjoint sets of players: the set of men M and the set of women W. A generic man is denoted as m, a woman as w, and a pair of a man and a woman as (m, w). A man might end up marrying some $w \in W$ or remaining single. All men's preferences over $W \cup \{s\}$, where s stands for being single, are represented by $u : M \times (W \cup \{s\}) \mapsto \mathbb{R}$ where $u(m, \cdot)$ is m's Bernoulli utility function over $W \cup \{s\}$. Likewise all women's preferences are represented by $v : (M \cup \{s\}) \times W \mapsto \mathbb{R}$ where $v(\cdot, w)$ is w's Bernoulli utility function over $M \cup \{s\}$. The **marriage market** (or simply **market**) is summarized by the tuple (M, W, u, v).

Let $\succeq_m$ denote man m's preference relation over $W \cup \{s\}$ induced by $u(m, \cdot)$: $w \succeq_m w'$ if and only if $u(m, w) \geq u(m, w')$. Likewise let $\succeq_w$ denote woman w's preference relation over $M \cup \{s\}$. Player y is **acceptable** to player x if $y \succ_x s$. A market is **trivial** if it does not have a mutually acceptable pair. A game starts with a market satisfying the following:

A1 Preferences are strict: $u(m, \cdot)$ is one-to-one for any $m \in M$; $v(\cdot, w)$ is one-to-one for any $w \in W$.
A2 Normalization: $u(m, s) = 0$ for any $m \in M$; $v(s, w) = 0$ for any $w \in W$.
A3 The market is finite: $|M| < \infty$ and $|W| < \infty$.
A4 The market is nontrivial.

2.2. The game rules

The game starts on day one ($t = 1$) with an **initial market** (M, W, u, v) and unfolds indefinitely into the future ($t = 2, 3, \ldots$). On each day a randomly selected pair $(m, w) \in M \times W$ meet. The random meeting process will be described in detail later. As they meet, m moves first to either accept or reject w. If m rejects w then the pair separate and return to the market. If m accepts w then it is w's turn to either accept or reject m. If w accepts m then (m, w) marry and leave the game for good; otherwise the pair separate and return to the market. Either the separation or the marriage concludes the current day. m and w receive one-time payoffs of $u(m, w)$

[5] Surveyed in Osborne and Rubinstein (1990) and Gale (2000).

and $v(m, w)$, respectively, upon marrying each other. The value of a marriage delayed by τ days is discounted by δ^τ where the common discount factor $\delta \in (0, 1)$ is meant to capture the overall search frictions. The game ends when there is no longer a mutually acceptable pair left in the market. A player receives a payoff of 0 when the game ends if he or she stays unmarried at that time.[6] Information is complete and past actions are perfectly observable.

2.3. Notations and terminology

Let H denote the set of all histories. Let $\hat{H}$ denote the set of all nonterminal histories after which a new day starts but the pair to meet on that day has not been determined. For $h \in H$ let $\Gamma(h)$ denote the subsequent subgame given h is reached. Note that $\Gamma(h)$ per se is a proper game if and only if $h \in \hat{H}$.

Let Z denote the set of all terminal histories. A terminal history may be infinite. The **outcome matching** of $h \in Z$ is a mapping $\mu_h : M \cup W \mapsto M \cup W \cup \{s\}$ such that $\mu_h(x)$ is player x's corresponding spouse if x managed to marry at some point along h, or otherwise $\mu_h(x) = s$. In the latter case x is said to be **single** under h. If h is finite then x is single if he or she stays unmarried until the game ends at h. If h is infinite then x is single if he or she is unmarried after any finite subhistory of h.

The market (M', W', u', v') is a **submarket** of the initial market (M, W, u, v) if $M' \subset M$, $W' \subset W$, u' is u restricted to $M' \times (W' \cup \{s\})$, and v' is v restricted to $(M' \cup \{s\}) \times W'$. Abuse notation to write (M', W', u, v) for simplicity. Let $\mathscr{S}$ denote the set of all nontrivial submarkets of the initial market. Given $S := (M', W', u, v)$, respectively use the notations $x \in S$ to denote $x \in M' \cup W'$, $(m, w) \in S$ to denote $(m, w) \in M' \times W'$, and $S\backslash(m, w)$ to denote the submarket $(M'\backslash\{m\}, W'\backslash\{w\}, u, v)$.

For $S \in \mathscr{S}$ and $x \in S$, let $A^S(x)$ denote the set $\{y \in S : y \succ_x s \text{ and } x \succ_y s\}$. $A^S(x)$ is thus the set of all players in S with whom x forms a mutually acceptable pair. Let $\alpha^S(x)$ denote the greatest element in $A^S(x) \cup \{s\}$ according to $\succeq_x$.

For $h \in H$ the **remaining market** after h, denoted as $S(h)$, consists of the men and women who are unmarried after h. Obviously $S(h) \in \mathscr{S}$ for any nonterminal $h \in H$.

2.4. The contact function

Recall that on each day a pair of a man and a woman are randomly selected to meet each other. The random meeting process is modeled by the **contact function** $C : M \times W \times \mathscr{S} \mapsto [0, 1]$, where $C(m, w, S)$ is the probability that (m, w) meet on a day at the beginning of which the remaining market is S. The game rules thus require that for any $S \in \mathscr{S}$,

B1 Only unmarried people meet: $C(m, w, S) = 0$ if $m \notin S$ or $w \notin S$.
B2 A meeting takes place on each day: $\sum_{(m,w)\in S} C(m, w, S) = 1$.

In addition, assume the meeting probability of any remaining pair is considerably large:

[6] It might be more natural to let the game end until no man or woman is left. Lemma 4.1 to appear later, which still holds under this alternative game-ending rule, implies under the alternative rule no one will marry and everyone's expected payoff is 0 in any subgame perfect equilibrium when the remaining market is trivial. Thus the default game-ending rule neither creates nor destroys equilibria in effect, yet it simplifies the exposition.

B3 There exists $\epsilon > 0$ such that $C(m, w, S) > \epsilon$ if $(m, w) \in S$.

Note that by the definition of the contact function, the meeting probabilities on a given day are determined by the remaining market at the beginning of that day. This implies that $\Gamma(h)$ and $\Gamma(h')$ are isomorphic for any histories h and h' such that $S(h) = S(h')$.

The game is summarized by the tuple (M, W, u, v, C, δ).

3. From equilibria to matchings

3.1. Equilibria

For most of the analysis the solution concept that will be applied is the subgame perfect equilibrium. In addition I consider two equilibrium selection criteria to accommodate more restrictive information settings.

For history h let $g(h)$ denote the sequence $(m_t, w_t, R_t)_{t=1:\tau(h)}$ where m_t and w_t are the man and woman who met on date t under h, $R_t \in \{marriage, separation\}$ is the result of that meeting, and $\tau(h)$ is the date of the last concluded meeting under h. A strategy profile σ satisfies the **private-dinner condition** if $g(h) = g(h')$ implies σ restricted to $\Gamma(h)$ is the same as σ restricted to $\Gamma(h')$. The private-dinner condition accommodates the information setting in which players are aware of who met whom in the past and the results of those meetings but not what happened during those meetings, presumably because the meetings took place over private dinners. In particular, if a meeting ended in separation there is no telling whether it was the man or the woman who said no. Note that the private-dinner condition implies a player's strategy cannot depend on actions taken during a failed meeting even if himself or herself participated in it.

A strategy profile satisfies the **Markov condition** if for any history h the player who moves after h conditions his or her behavior only on $S(h)$. The Markov condition is stronger than the private-dinner condition because $g(h) = g(h')$ implies $S(h) = S(h')$ but not vice versa. The Markov condition is compatible with the more restrictive information setting in which players are only aware of the current market.

When describing strategies, I will simply say "player x accepts/rejects player y under condition K" to represent the statement that x accepts/rejects y at every decision point satisfying condition K where it is x's turn to make the pertinent decision. I say (m, w) **marry upon first meeting** under strategy profile σ if on the equilibrium path the first meeting between (m, w) results in marriage. (m, w) marry upon first meeting if and only if on the equilibrium path (m, w) always accept each other.

3.2. Matchings

A **matching** for (M, W, u, v) is a scheme that pairs some players into married couples and leaves others single. A matching is formalized as a function $\mu : M \cup W \mapsto M \cup W \cup \{s\}$ such that $\mu(x) \in W \cup \{s\}$ if $x \in M$, $\mu(x) \in M \cup \{s\}$ if $x \in W$, and $\mu(\mu(x)) = x$ if $\mu(x) \neq s$. μ is **unstable** if there is a player x such that $s \succ_x \mu(x)$, in which case μ is **individually blocked** by x, or if there is a pair (m, w) such that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$, in which case μ is **pairwise blocked** by (m, w). μ is **stable** if it is not unstable. Gale and Shapley (1962) shows that at least one stable matching exists for any marriage market, and moreover there is a **men-optimal matching** commonly agreed by all men as the best stable matching and likewise there is a **women-optimal matching**. Given a matching μ for the market (M, W, u, v) let $\mathscr{S}_\mu$ denote the set of all nontrivial submarkets (M', W', u, v) such that $W \backslash W' = \mu(M \backslash M')$ where $\mu(M \backslash M')$ denotes the μ-image of $M \backslash M'$. Observe that if $S \in \mathscr{S}_\mu$ then μ restricted to S is a matching for S; moreover if μ is a stable matching for (M, W, u, v) then μ restricted to S is a stable matching for S.

Obviously the outcome matching of any $h \in Z$ is a matching for the initial market. A strategy profile σ and the contact function C jointly induce a probability measure on 2^Z and hence also induce a probability mass function on the set of all matchings for the initial market. We say that a matching **obtains** if it arises as an outcome matching. A strategy profile σ **enforces** a matching μ if μ obtains almost surely under σ. μ being enforced implies the players will almost surely be coupled together or left single according to μ.

3.3. Near-frictionless analysis

This paper focuses on analyzing game outcomes when search frictions are small. With respect to this approach we introduce the following terminology: An **environment** $(M, W, u, v, C) := \{(M, W, u, v, C, \delta) : \delta \in (0, 1)\}$ is the set of all games that share the same initial market and contact function. A strategy profile σ is a **limit equilibrium** of the environment (M, W, u, v, C) if there exists some $d < 1$ such that σ is a subgame perfect equilibrium of the game (M, W, u, v, C, δ) for any $\delta > d$.

4. Analysis

4.1. Preliminary results

The following lemma collects some useful results for future reference.

Lemma 4.1. *For a subgame perfect equilibrium σ let $\pi(x)$ denote the expected payoff for player x under σ. The following are true for σ:*

(a) $\pi(x) \geq 0$ for any $x \in M \cup W$.
(b) (m, w) marry with positive probability only if m is acceptable to w.[7]
(c) (m, w) marry with positive probability only if $\alpha^{S_I}(m) \succeq_m w$ and $\alpha^{S_I}(w) \succeq_w m$ where S_I denotes the initial market.
(d) $\pi(m) \leq u(m, \alpha^{S_I}(m))$ for any $m \in M$. $\pi(w) \leq v(\alpha^{S_I}(w), w)$ for any $w \in W$.

Proof. (a) follows from the observation that a player secures an expected payoff of 0 by rejecting everyone forever. The same observation implies a woman's equilibrium continuation payoff from rejecting a man is nonnegative, thus (b) follows. To show (c), first observe that if $w \succ_m \alpha^{S_I}(m)$ then m is unacceptable to w, thus (m, w) will not marry by (b). Suppose $m \succ_w \alpha^{S_I}(w)$ yet (m, w) marry with positive probability. w is unacceptable to m. That (m, w) marry with positive probability implies m accepts w with positive probability after some history h. Let h' denote the immediate history following h as m has accepted w. w rejects m with positive probability after h' because otherwise m's expected payoff from accepting w after h is $u(m, w) < 0$, less than the payoff of 0 from rejecting everyone forever. Let V denote w's expected payoff in the subsequent

[7] However, a pair (m, w), where w is unacceptable to m, might marry in equilibrium with positive probability. For an example of this possibility see Appendix A.3. Before reading the example the reader is recommended to go through Proposition 4.2 and Example 1 in the main text.

subgame Γ as she has rejected m after h'. That w rejects m with positive probability after h' implies $\delta V \geq v(m, w)$, then in turn implies there exists some $m' \succ_w m$ such that (m', w) marry with positive probability in Γ. Since $m' \succ_w \alpha^{S_I}(w)$, we can apply the same argument for m' and conclude there exists some m'' such that $m'' \succ_w m'$ and (m'', w) marry with positive probability in some subgame. Iteratively applying the same argument leads to the necessary contradiction because M is finite. (d) follows from (c). □

4.2. *Enforcing stable matchings*

The foremost question of whether stable matchings may be enforced in equilibria is addressed in this subsection. Proposition 4.2 below gives a positive answer by showing that any stable matching can be enforced in a limit equilibrium. The proof is based on a construction per se worth highlighting: For a matching μ of the initial market, the **μ-strategy profile** σ_μ is described by the following table that specifies what man m and woman w do if they meet each other when the remaining market is S. In the table μ^S denotes the women-optimal matching for S.

	m	w
If $S \in \mathscr{S}_\mu$	Accept w if $w \succeq_m \mu(m)$	Accept m if $m \succeq_w \mu(w)$
If $S \notin \mathscr{S}_\mu$	Accept w if $w \succeq_m \mu^S(m)$	Accept m if $m \succeq_w \mu^S(w)$

Let $V_\mu(x, \delta)$ be player x's expected payoff under σ_μ if the discount factor is δ.

Proposition 4.2. *If μ is a stable matching for the initial market of the environment (M, W, u, v, C) then:*

(a) σ_μ enforces μ.
(b) (m, w) marry upon first meeting under σ_μ if $w = \mu(m)$.
(c) $\lim_{\delta\to 1} V_\mu(m, \delta) = u(m, \mu(m))$ and $\lim_{\delta\to 1} V_\mu(w, \delta) = v(\mu(w), w)$.
(d) σ_μ satisfies the Markov condition.
(e) σ_μ is a limit equilibrium of (M, W, u, v, C).

Proof. The initial market is in $\mathscr{S}_\mu$. Observe that if (m, w) meet when the remaining market is in $\mathscr{S}_\mu$ then the meeting results in marriage if and only if $w = \mu(m)$ because μ being stable implies $w \succeq_m \mu(m)$ and $m \succeq_w \mu(w)$ hold simultaneously if and only if $w = \mu(m)$. Thus the remaining market after any history on the equilibrium path is in $\mathscr{S}_\mu$, which combined with the previous observation implies (b). It also follows that (m, w) will not marry on the equilibrium path if $w \neq \mu(m)$. Consequently if $\mu(m) \neq s$ then $(m, \mu(m))$ remain in the market until the first meeting between them takes place. Thus the probability that $(m, \mu(m))$ marry is equal to the probability that they meet eventually, the latter being bounded from below by $\sum_{n=0}^{\infty} \epsilon(1-\epsilon)^n = 1$, implying (a). Following (a) and (b) we have $u(m, \mu(m)) \geq V_\mu(m, \delta) \geq \sum_{n=0}^{\infty} \epsilon[\delta(1-\epsilon)]^n u(m, \mu(m))$. (c) follows from $\lim_{\delta\to 1} \sum_{n=0}^{\infty} \epsilon[\delta(1-\epsilon)]^n u(m, \mu(m)) = u(m, \mu(m))$ and the analogous equality for any w. (d) follows from the observation that a player's behavior depends on only the current remaining market.

Now show (e). Suppose (m, w) meet on a day when the remaining market is S. Let $\overline{\mu}$ denote μ restricted to S if $S \in \mathscr{S}_\mu$ or μ^S otherwise. Thus $\overline{\mu}$ is a stable matching for S. Let Γ denote the subsequent subgame resulting from (m, w)'s separation. By construction σ_μ restricted to Γ is equal to the $\overline{\mu}$-strategy profile $\sigma_{\overline{\mu}}$ of Γ. Let $V_\mu(x, \delta|\Gamma)$ denote the expected payoff for player $x \in S$ under σ_μ restricted to Γ. Then $\lim_{\delta\to 1} \delta V_\mu(x, \delta|\Gamma) = u(m, \overline{\mu}(m))$ by (c) and $\delta V_\mu(x, \delta|\Gamma) < u(m, \overline{\mu}(m))$ by (a); the analogous equality and inequality respectively hold for w. Apply one-deviation analysis for m and w. For δ sufficiently close to 1, $\delta V_\mu(w, \delta|\Gamma) < v(m, w)$ if and only if $m \succeq_w \overline{\mu}(w)$, where the left side of the inequality is w's expected payoff from rejecting m and the right side that from accepting m. Thus accepting m if and only if $m \succeq_w \overline{\mu}(w)$ is optimal for w. m's expected payoff from rejecting w is $\delta V_\mu(m, \delta|\Gamma)$ whereas that from accepting w is $u(m, \overline{\mu}(m))$ if $w = \overline{\mu}(m)$, $\delta V_\mu(m, \delta|\Gamma)$ if $w \succ_m \overline{\mu}(m)$, or $pu(m, w) + (1 - p)\delta V_\mu(m, \delta|\Gamma)$ if $\overline{\mu}(m) \succ_m w$ where p is either 0 or 1. Thus accepting w if and only if $w \succeq_m \overline{\mu}(m)$ is optimal for m if δ is sufficiently close to 1. $\square$

Proposition 4.2 agrees with the common finding that the core of a coalitional game can be supported in equilibria of a non-cooperative counterpart. For results along this line in a similar search and matching context see Rubinstein and Wolinsky (1990), Adachi (2003), and Lauermann and Nöldeke (2014). The limit equilibria constructed above will also be used as important building blocks for more complicated equilibria.

4.3. *Enforcing unstable matchings*

In this subsection, the question of whether unstable matchings are enforceable in limit equilibria is addressed with an affirmative example.

Example 1 *(Reward and punishment).* To describe the initial market, player x's preferences are represented by a list $P(x)$ such that $P(x) = a, \dots, b$ if and only if $a \succ_x \dots \succ_x b \succ_x s$. Note that players unacceptable to x are omitted from $P(x)$. The initial market is represented as

$$\begin{aligned} &P(m_1) = w_2, w_1, && P(w_1) = m_1, m_2, m_3 \\ &P(m_2) = w_1, w_2, && P(w_2) = m_2, m_3, m_1, \\ &P(m_3) = w_2. && \end{aligned}$$

A limit equilibrium σ is constructed to enforce μ such that $\mu(m_1) = w_2$, $\mu(m_2) = w_1$ and $\mu(m_3) = s$. μ is unstable because the pair (m_3, w_2) blocks it. σ is specified by an automaton with the following states:

q_0: The initial state. In q_0, m_1 accepts w_2; m_2 accepts no one; m_3 accepts no one; w_1 accepts m_1; w_2 accepts m_1 and m_2. The transition rules are:

$$q_0 \longrightarrow \begin{cases} q_1 & \text{if } (m_1, w_2) \text{ marry,} \\ q_2 & \text{if for some } (m, w) \neq (m_1, w_2)\text{: } w \text{ rejects } m \text{ or } (m, w) \text{ marry,} \\ q_0 & \text{otherwise.} \end{cases}$$

q_1: An absorbing state. As the state has just become q_1, the remaining market is $S_1 := S_0 \backslash (m_1, w_2) \in \mathscr{S}_\mu$ where S_0 denotes the initial market. In q_1 the players follow the μ^{S_1}-strategy profile where μ^{S_1} is μ restricted to S_1.

q_2: An absorbing state. Let S_2 denote the (history-dependent) remaining market as the state has just become q_2. In q_2 the players follow the μ^{S_2}-strategy profile where μ^{S_2} denotes the women-optimal matching for S_2.

On the equilibrium path (m_1, w_2) marry first. Then the state becomes q_1 in which μ^{S_1} (μ restricted to S_1) is enforced, as implied by Proposition 4.2(a), because μ^{S_1} is a stable matching for S_1. Thus μ obtains under any finite terminal history on the equilibrium path. It is straightforward to verify that the game ends almost surely, implying μ obtains almost surely. Thus σ enforces μ.

Now verify that σ is indeed a limit equilibrium. By Proposition 4.2(e), σ restricted to subgames in q_1 and q_2 is a limit equilibrium of the respective subgames. Thus it suffices to check q_0. Consider the situation that the blocking pair (m_3, w_2) meet on a day in q_0. Apply one-deviation analysis. Suppose m_3 has accepted w_2. w_2's action of rejecting m_3 will switch the state to q_2 in which the μ^W-strategy profile will be implemented where μ^W is the women-optimal matching for the initial market. By Proposition 4.2(c), w_2's expected payoff from rejecting m_3 is approximately $v(\mu^W(w_2), w_2) = v(m_2, w_2)$ for δ sufficiently close to 1, strictly greater than $v(m_3, w_2)$. Thus rejecting m_3 is optimal when near-frictionless. Now consider m_3. If he rejects w_2 then μ is enforced; otherwise if he accepts w_2 then w_2 will reject him, switching the state to q_2 in which μ^W is enforced. $\mu(m_3) = \mu^W(m_3) = s$ implies rejecting w_2 is (weakly) optimal for m_3.

Consider the situation that (m_2, w_1) meet in q_0. As in w_2's case above, it is optimal for w_1 to reject m_2 for δ sufficiently close to 1. m_2's case is slightly different from m_3's case above. If m_2 accepts w_1, the μ^W-strategy profile will be implemented under which m_2's expected payoff is approximately $u(m, \mu^W(m)) = u(m_2, w_2)$ for δ sufficiently close to 1. If m_2 rejects w_1, m_2 will marry w_1 eventually but only after (m_1, w_2) marry. A lower bound for m_2's expected payoff from rejecting w_1 can thus be computed as $\left[\frac{\delta\epsilon}{1-\delta(1-\epsilon)}\right]^2 u(m_2, w_1)$, strictly greater than $u(m_2, w_2)$ for δ sufficiently close to 1. Rejecting w_1 in q_0 is optimal when near-frictionless. The optimality of σ in other cases is either similar to those discussed above or can be verified by routine inspection. □

A blocking pair would profit from marrying each other to circumvent an unstable matching. To enforce an unstable matching such circumvention must be discouraged. In Example 1, a reward-punishment scheme, implemented in q_2, is employed to prevent the blocking attempt from (m_3, w_2). To see the point, note that if m_3 initiated a blocking attempt by accepting w_2, w_2 would not oblige because she would receive a reward, which is the promise of marrying the more preferable man m_2, from rejecting m_3. In contrast m_2 would be (weakly) punished[8] for initiating the blocking attempt by being forced to stay single. Meanwhile, to ensure the reward for w_2 is credible, m_2 needs to be available until either m_3 or w_2 has married. In Example 1, m_2 may marry only after (m_1, w_2) have married. m_2's potential attempt to marry w_1 early is discouraged by a similar reward-punishment scheme. Should m_2 accept w_1 when (m_1, w_2) have not married, he would be strictly punished (by marrying w_2 eventually) for deviating and w_1 would be rewarded (by marrying m_1 eventually) for not obliging. The reward-punishment schemes resemble those used in Proposition 1 in Rubinstein and Wolinsky (1990) supporting non-core outcomes. In their model, a reward-punishment scheme targeted at the blocking attempt between a buyer and a seller entails reaction from at most three players (those whose welfare would be affected should the attempt succeed), because all sellers are identical and so are all buyers. In contrast, in the present model, because of a more complicated preference structure, a reward-punishment scheme may require the entire market to re-coordinate, which would make its implementability

[8] Example 1 relies on the knife-edge case that m_3 rejects w_2 when indifferent. Such fragility need not be present in enforcing an unstable matching. In an earlier version of this paper I provided a more complicated example in which all circumventing attempts are strictly punished.

more difficult. Indeed, Proposition 4.6 to appear later will show for certain markets no unstable matching can be enforced.

4.4. Sufficient conditions for enforced matchings to be stable

In this subsection two points are made regarding the enforceability of unstable matchings. First, disabling reward-punishment schemes excludes unstable matchings from matchings enforceable in equilibria. Second, enforceability of unstable matchings depends on the preference structure. Each point is made by a condition under which all matchings enforceable in equilibria are stable.

The following lemma will be useful for the proofs of further results.

Lemma 4.3. *If an unstable matching μ is enforced in a subgame perfect equilibrium σ then for any pair (m, w) blocking μ, σ prescribes the following for any on-equilibrium-path meeting between (m, w): m rejects w, moreover if m deviated to accepting w then w would reject m with positive probability.*

Proof. Consider any on-equilibrium-path meeting between a blocking pair (m, w). Suppose w would accept m with probability 1 after she has been accepted by m. m must reject w in the first place, because otherwise (m, w) would marry with positive probability in equilibrium, a contradiction. It follows that m's continuation payoff from rejecting w is weakly higher than $u(m, w)$, the latter strictly higher than $u(m, \mu(m))$ since the pair (m, w) blocks μ, implying m would marry someone other than $\mu(m)$ with positive probability in equilibrium, a contradiction. Thus w's equilibrium strategy is to reject m with positive probability in the current meeting. This implies w's expected payoff from rejecting m is weakly greater than $v(m, w)$, the latter strictly greater than $v(\mu(w), w)$. Hence in the subsequent subgame Γ resulting from w having rejected m, w will marry with someone other than $\mu(w)$ with positive probability. Suppose m accepts w with positive probability in the current meeting, then Γ is reached with positive probability, implying w would marry someone other than $\mu(w)$ with positive probability in equilibrium, a contradiction. $\square$

Lemma 4.3 implies that in order to deter a blocking pair (m, w) from circumventing the enforcement of an unstable matching, m must be punished for initiating a blocking attempt and w rewarded for not obliging. Obviously such a scheme cannot be implemented if the other players cannot tell whether (m, w)'s separation resulted from a failed blocking attempt initiated by m but turned down by w, or from m having rejected w as prescribed. As the upcoming proposition shows, the private-dinner condition, which disallows the players from conditioning their behavior on the actions during any failed meeting, indeed rules out unstable matchings from matchings enforceable in equilibria.

Proposition 4.4. *No unstable matching can be enforced in a private-dinner equilibrium.*

Proof. Prove by contradiction. Suppose an unstable matching μ for the initial market is enforced in a private-dinner equilibrium σ. μ is individually rational by Lemma 4.1(a), thus some pair (m, w) blocks μ. Consider the situation that (m, w) meet on the first day of the game. Let $h \in \hat{H}$ denote the history corresponding to m having rejected w on the current day. Let $h' \in \hat{H}$ denote the history corresponding to m having accepted w and w having rejected m on the current day. Thus $g(h)$ and $g(h')$ are both the one-entry sequence $(m, w, \mathit{separation})$. Let V and V' respectively denote w's expected payoffs in $\Gamma(h)$ and $\Gamma(h')$. The private-dinner condition implies $V = V'$. By Lemma 4.3, w would reject m with positive probability on the current day if she was accepted, implying $\delta V' \geq v(m, w)$. h is on the equilibrium path by Lemma 4.3 because m rejects w on the current day in equilibrium. Thus μ is enforced by σ restricted to $\Gamma(h)$, implying $v(\mu(w), w) \geq V$. Since the pair (m, w) blocks μ, we have $v(m, w) > v(\mu(w), w)$, implying $\delta V = \delta V' \geq v(m, w) > v(\mu(w), w) \geq V$, a contradiction. □

Proposition 4.4 identifies a condition on the equilibrium that negates enforceability of unstable matchings *for any initial market*. In contrast the upcoming proposition identifies a condition on the initial market that negates enforceability of unstable matchings *in any equilibrium*.

Call (m, w) a **top pair** for submarket S if $m = \alpha^S(w)$ and $w = \alpha^S(m)$. (m, w) is a top pair for S if for m, w is the best woman among those in S who find m acceptable, and vice versa. A marriage market satisfies the **Sequential Preference Condition** if there is an ordering of the men $m_1, \ldots, m_{|M|}$, an ordering of the women $w_1, \ldots, w_{|W|}$, and a positive integer k such that:

1. For any $i \leq k$, (m_i, w_i) is a top pair for $S_i := (\{m_j : j \geq i\}, \{w_j : j \geq i\}, u, v)$.
2. $S_{k+1} := (\{m_j : j \geq k+1\}, \{w_j : j \geq k+1\}, u, v)$ is trivial.

A stronger condition, introduced in Eeckhout (2000), implies that the market has a unique stable matching.[9] The present Sequential Preference Condition, albeit weaker, is still sufficient for uniqueness. The unique stable matching pairs m_i to w_i for any $i \leq k$ and leaves m_i and w_i single for any $i > k$. The Sequential Preference Condition implies that the players' preferences are aligned in a certain way, see Eeckhout (2000) for a different formalization that highlights the alignment.

Lemma 4.5. *If (m, w) is a top pair for the initial market then (m, w) marry with positive probability, and upon first meeting, in any subgame perfect equilibrium.*

Proof. If (m, w) is a top pair for the initial market then clearly they are a top pair for any submarket in which both are present. w always accepts m in any subgame perfect equilibrium because the equilibrium payoff from rejecting m is strictly less than $v(m, w)$ by Lemma 4.1(d). It follows that in any subgame perfect equilibrium m always accepts w because his payoff from accepting w is $u(m, w)$ whereas that from rejecting w is strictly less than $u(m, w)$ by Lemma 4.1(d). Thus (m, w) marry upon first meeting in any subgame perfect equilibrium. That they marry with positive probability follows from the assumption that they meet on the first day of the game with positive probability. □

Proposition 4.6. *If the initial market satisfies the Sequential Preference Condition then no unstable matching can be enforced in any subgame perfect equilibrium.*

Proof. Let the men and women be ordered as in the definition of the Sequential Preference Condition and k be the corresponding index threshold. Fix a subgame perfect equilibrium σ that

[9] The condition in Eeckhout (2000), sometimes also called the Sequential Preference Condition in the literature, is equivalent to the present condition if $|M| = |W|$, every man is acceptable to every woman and vice versa.

enforces some matching μ. (m_1, w_1) marry with positive probability under σ by Lemma 4.5. It follows that $\mu(m_1) = w_1$. It also follows that a subgame with remaining market S_2 is reached with positive probability, because if (m_1, w_1) meet on the first day, which occurs with positive probability, then they marry and the remaining market becomes S_2. Suppose for any $i < n \le k$ for some n, $\mu(m_i) = w_i$ and a subgame with remaining market S_{i+1} is reached with positive probability. By the inductive hypothesis a subgame Γ with remaining market S_n is reached with positive probability. By Lemma 4.5, (m_n, w_n) marry with positive probability in Γ under σ, implying that (m_n, w_n) marry with positive probability in the initial game under σ. Hence $\mu(m_n) = w_n$. It also follows that a subgame with remaining market S_{n+1} is reached with positive probability. By induction, for any $i \le k$, $\mu(m_i) = w_i$ and a subgame with remaining market S_{i+1} is reached with positive probability, implying the subgame with remaining market S_{k+1} is reached with positive probability. As the remaining market becomes S_{k+1} the game ends and any player $x \in S_{k+1}$ remains single. Thus $\mu(m_i) = \mu(w_i) = s$ for any $i > k$. The proposition follows from the observation that μ is the unique stable matching. □

For the sequential bargaining model considered in Suh and Wen (2008), if the underlying marriage market satisfies the Sequential Preference Condition then there is a unique equilibrium implementing the unique stable matching (Theorem 1).[10] However, the Sequential Preference Condition does *not* guarantee a unique subgame perfect equilibrium for the present model, as will be shown in a later example (Example 3). In some of the additional equilibria, a lottery of matchings, instead of a single matching, is induced due to the uncertainty in the search process, which is absent from the model in Suh and Wen (2008).

4.5. Equilibria inducing a lottery of matchings

Instead of enforcing a single matching, there may exist equilibria inducing a lottery of multiple matchings. In this subsection examples are given showing that unstable matchings, even Pareto-dominated ones, may obtain with positive probability in such equilibria.

Example 2 *(Regret).* The initial market is described by the following lists:

$$
\begin{aligned}
&P(m_1) = w_2, w_1, w_3, && P(w_1) = m_1, m_2, m_3,\\
&P(m_2) = w_3, w_2, w_1, && P(w_2) = m_2, m_1, m_3,\\
&P(m_3) = w_2, w_3, w_1, && P(w_3) = m_3, m_2, m_1,\\
&P(m_1') = w_2', w_1', w_3', && P(w_1') = m_1', m_2', m_3',\\
&P(m_2') = w_3', w_2', w_1', && P(w_2') = m_2', m_1', m_3',\\
&P(m_3') = w_2', w_3', w_1', && P(w_3') = m_3', m_2', m_1'.
\end{aligned}
$$

Moreover, $\frac{1}{2}v(m_2, w_2) + \frac{1}{2}v(m_3, w_2) > v(m_1, w_2)$ and $\frac{1}{2}v(m_2', w_2') + \frac{1}{2}v(m_3', w_2') > v(m_1', w_2')$. The contact function satisfies $C(m, w, S) = C(\hat{m}, \hat{w}, S)$ for any pairs $(m, w) \in S$ and $(\hat{m}, \hat{w}) \in S$ for any $S \in \mathscr{S}$. The initial market has a unique stable matching μ such that $\mu(m_i) = w_i$ and $\mu(m_i') = w_i'$ for $i = 1, 2, 3$.

[10] Theorem 1 in Suh and Wen (2008) uses the stronger condition from Eeckhout (2000), but remains true under the present Sequential Preference Condition.

In the limit equilibrium σ to be described shortly, two matchings, μ_a and μ_b given below, obtain, each with probability 0.5:

$$\mu_a(m_1) = w_1, \quad \mu_a(m_2) = w_3, \quad \mu_a(m_3) = w_2, \quad \mu_a(m_i') = w_i', i = 1, 2, 3,$$
$$\mu_b(m_1') = w_1', \quad \mu_b(m_2') = w_3', \quad \mu_b(m_3') = w_2', \quad \mu_b(m_i) = w_i, i = 1, 2, 3.$$

Neither μ_a nor μ_b is stable: the pair (m_1, w_2) blocks μ_a and the pair (m_1', w_2') blocks μ_b. σ is specified by an automaton with the following states:

q_0: The initial state. m_1 accepts w_1 and w_2. m_1' accepts w_1' and w_2'. Every other man accepts no one. Every woman accepts her first choice. The transition rules are

$$q_0 \longrightarrow \begin{cases} q_a & \text{if } (m_1, w_1) \text{ marry}, \\ q_b & \text{if } (m_1', w_1') \text{ marry}, \\ q_3 & \text{if some couple other than } (m_1, w_1) \text{ or } (m_1', w_1') \text{ marry}, \\ q_0 & \text{otherwise}. \end{cases}$$

q_a: An absorbing state. As the state has just become q_a the remaining market is $S_a := S_0 \backslash (m_1, w_1) \in \mathscr{S}_{\mu_a}$ where S_0 denotes the initial market. In q_a the players follow the $\hat{\mu}_a$-strategy profile where $\hat{\mu}_a$ is μ_a restricted to S_a.

q_b: An absorbing state. As the state has just become q_b the remaining market is $S_b := S_0 \backslash (m_1', w_1') \in \mathscr{S}_{\mu_b}$. In q_b the players follow the $\hat{\mu}_b$-strategy profile where $\hat{\mu}_b$ is μ_b restricted to S_b.

q_3: An absorbing state. Let S_3 denote the (history-dependent) remaining market as the state has just become q_3. In q_3 the players follow the μ^{S_3}-strategy profile where μ^{S_3} denotes the women-optimal matching for S_3.

Note that σ satisfies the private-dinner condition. μ_a obtains almost surely conditional on q_a being reached. μ_b obtains almost surely conditional on q_b being reached. q_a is reached if the first meeting between (m_1, w_1) takes place before the first meeting between (m_1', w_1'); q_b is reached if the first meeting between (m_1', w_1') takes place before the first meeting between (m_1, w_1). It is easy to verify that μ_a and μ_b both obtain with probability 0.5.

To see that σ is indeed a limit equilibrium it suffices to check the absence of profitable one-deviations in q_0 because μ_a restricted to S_a, μ_b restricted to S_b, and μ^{S_3} restricted to S_3 are stable matchings for the respective submarkets and hence Proposition 4.2 is applicable to these cases. Let $V(x, \delta)$ denote the expected payoff for player x under σ if the discount factor is δ. For any $h \in \hat{H}$ such that the state of h is q_0, player x's expected payoff in $\Gamma(h)$ is also $V(x, \delta)$. We have $\overline{V}(m) := \lim_{\delta \to 1} \delta V(m, \delta) = \frac{1}{2}u(m, \mu_a(m)) + \frac{1}{2}u(m, \mu_b(m))$ for any $m \in M$ and $\overline{V}(w) := \lim_{\delta \to 1} \delta V(w, \delta) = \frac{1}{2}v(\mu_a(w), w) + \frac{1}{2}v(\mu_b(w), w)$ for any $w \in W$. Observe that $v(m, w) > \overline{V}(w)$ if and only if m is w's first choice. Thus accepting only her first choice is optimal for every woman in state q_0 given δ sufficiently close to 1. In particular it is optimal for w_2 to reject m_1 despite they form a blocking pair against μ_a and for w_2' to reject m_1' despite they form a blocking pair against μ_b. The men's incentives can be verified by routine inspection. □

Example 2 shows that, despite Proposition 4.4, unstable matchings may obtain in private-dinner equilibria. Suppose (m_1, w_2), a pair that blocks one of the probable outcome matchings μ_a, meet when the remaining market is the initial market. At this point (m_1, w_2) do not both profit from circumventing the equilibrium by marrying each other, because the game can still go either way, leading to μ_a or μ_b. Indeed, m_1 would accept w_2 yet w_2 would reject m_1, because w_2 still has a chance to marry a more preferred man, m_2, with probability 0.5. If (m_1', w_1') is the first couple to marry then w_2 will marry m_2 eventually. However, if instead (m_1, w_1) is the first couple to marry, then for w_2, m_2 is impossible and m_1 is no more, forcing her to marry the last choice m_3. w_2 would regret that she had rejected m_1, as an old English saying goes: *He that will not when he may; when he will, he shall have Nay.*

Regret occurs in a nonstationary setting because the search prospect may change as the game unfolds. Regret does not occur in a model with a completely stationary search setting, such as the one in Adachi (2003). The model in Lauermann and Nöldeke (2014) also assumes a stationary search setting but has a trace of nonstationarity: Players may be forced to leave the market as singles, at the point of which one's continuation payoff drops to zero. It is this trace of nonstationarity that makes possible the emergence of regret and unstable matchings in some equilibria in which players are forced to leave as singles after having rejected acceptable options in the past.

Example 3 *(Coordination failure).* The initial market is described by the following lists:

$$P(m_1) = w_1, w_2, \qquad P(w_1) = m_1, m_3,$$
$$P(m_2) = w_2, w_3, \qquad P(w_2) = m_2, m_1,$$
$$P(m_3) = w_3, w_1, \qquad P(w_3) = m_3, m_2.$$

Moreover, for any $m \in M$, if w and w' are m's first and second choices respectively then $u(m, w') > \frac{2}{3}u(m, w) + \frac{1}{6}u(m, w')$. Similarly for any $w \in W$, if m and m' are w's first and second choices respectively then $v(m', w) > \frac{2}{3}v(m, w) + \frac{1}{6}v(m', w)$. The contact function satisfies $C(m, w, S) = C(\hat{m}, \hat{w}, S)$ for any pairs $(m, w) \in S$ and $(\hat{m}, \hat{w}) \in S$ for any $S \in \mathscr{S}$. Observe that the initial market satisfies the Sequential Preference Condition and so do all of its submarkets.

In the limit equilibrium σ, when the remaining market is the initial market, each man accepts every acceptable woman and each woman accepts every acceptable man. After the first marriage is realized the players follow the μ^S-strategy profile where μ^S is the unique stable matching for the remaining market S right after the first marriage. σ satisfies the Markov condition.

The following four matchings obtain with positive probability:

$$\begin{array}{llll} \mu_0: \; m_1 \mapsto w_1, & \mu_1: \; m_1 \mapsto w_2, & \mu_2: \; m_1 \mapsto w_1, & \mu_3: \; m_1 \mapsto s, \\ m_2 \mapsto w_2, & m_2 \mapsto s, & m_2 \mapsto w_3, & m_2 \mapsto w_2, \\ m_3 \mapsto w_3, & m_3 \mapsto w_3, & m_3 \mapsto s, & m_3 \mapsto w_1, \\ & w_1 \mapsto s, & w_2 \mapsto s, & w_3 \mapsto s. \end{array}$$

μ_0 is the unique stable matching for the initial market. Each of μ_i, $i = 1, 2, 3$ is Pareto-dominated by μ_0. It is easy to verify the following: μ_0 obtains almost surely conditional on the event E_0 that a man meets his first choice on the first day of the game. μ_i, $i = 1, 2, 3$, obtains almost surely conditional on the event E_i that m_i meets his second choice on the first day of the game. Let $p(\mu)$ denote the unconditional probability that μ obtains under σ. For any i,

$$p(\mu_i) = \Pr(E_i) \times 1 + \sum\nolimits_{j \neq i} \Pr(E_j) \times 0 + \Big(1 - \sum\nolimits_{j=0}^{3} \Pr(E_j)\Big) p(\mu_i).$$

To see that the equality holds, note that conditional on none of E_i, $i = 0, 1, 2, 3$, having occurred on the first day, the probability that μ_i obtains remains $p(\mu_i)$ because σ satisfies the Markov condition. By the specification of the contact function we have $\Pr(E_0) = 1/3$ and $\Pr(E_i) = 1/9$ for $i = 1, 2, 3$. Thus $p(\mu_0) = 1/2$ and $p(\mu_i) = 1/6$ for $i = 1, 2, 3$. Observe that under σ each player marries his or her first choice with probability 2/3, second choice with probability 1/3, and stays single with probability 1/6.

To verify that σ is indeed a limit equilibrium, it suffices to check the absence of profitable one-deviations when the remaining market is the initial market, since Proposition 4.2 covers the other cases. Let $V(x,\delta)$ denote player x's expected payoff under σ if the discount factor is δ. $V(x,\delta)$ is also x's expected payoff in $\Gamma(h)$ for any $h \in \hat{H}$ such that $S(h)$ is the initial market since σ satisfies the Markov condition. It is easy to verify that $\overline{V}(w) := \lim_{\delta \to 1} V(w,\delta) = \frac{2}{3}v(m,w) + \frac{1}{6}v(m',w)$ where m and m' are w's first and second choices respectively. We have $\delta V(w,\delta) < \overline{V}(w) < v(m',w)$ where the second inequality is by assumption, implying it is optimal for w to accept both m and m'. Similarly it is optimal for each man to accept every acceptable woman. $\square$

In Example 3, σ is strictly Pareto-dominated by the μ_0-strategy profile, which by Proposition 4.2 is also a limit equilibrium. Inefficiency arises because of a coordination failure due to self-confirmation of mutual doubts. Despite being each other's first choice, m_1 does not commit to waiting for w_1 because w_1 does not commit to waiting for m_1 because m_1 does not commit to waiting for w_1 and so on *ad infinitum*. The example shows that the Markov condition, which is stronger than the private-dinner condition, and the Sequential Preference Condition combined are not sufficient to rule out limit equilibria in which unstable matchings obtain with positive probability.

4.6. Delay

This subsection studies whether equilibrium delay may cause significant efficiency loss even if search frictions are small. Two types of delay come to mind. The first type refers to the situation that a game never ends. Recall a game ends when the remaining market becomes trivial. A never-ending game implies at least one mutually beneficial marriage is not realized while the pertained players stay unmarried into the infinite future. Such a situation is not unlike a never-ending negotiation over how to split the money on the table. The upcoming proposition essentially rules out this type of delay in equilibrium.

Proposition 4.7. *A game ends almost surely in any subgame perfect equilibrium.*

The proof, found in Appendix A.1, hinges on the observation that in any subgame perfect equilibrium, after any nonterminal history, the probability that no marriage occurs during the next T days is less than some constant $\overline{p} < 1$ if T is sufficiently large. It follows that the probability that no marriage occurs for a duration of kT days is less than $\overline{p}^k$, thus the probability that a marriage will occur during the next kT days becomes arbitrarily close to 1 as k tends to ∞, implying the initial market will eventually shrink to a trivial market.

In contrast to a never-ending game, the second type of delay is in its literal sense: Some marriages are realized too late. We define what it means to be "too late" as follows: Let $\mathscr{M}$ denote the set of all matchings for the initial market. For any strategy profile σ and man m define the efficiency loss due to delay under σ as

$$L_\sigma(m,\delta) := \sum\nolimits_{\mu \in \mathscr{M}} p_\sigma(\mu) u(m,\mu(m)) - V_\sigma(m,\delta)$$

where $p_\sigma(\mu)$ denotes the probability that μ obtains under σ and $V_\sigma(m,\delta)$ denotes m's expected payoff under σ if the discount factor is δ. Define $L_\sigma(w,\delta)$ for each woman w analogously. Efficiency loss is thus measured as the difference between a player's expected payoff from the immediate resolution of the lottery induced by σ and his or her expected payoff under σ. $\delta < 1$ implies $L_\sigma(x, \delta) \geq 0$. We ask whether equilibrium efficiency loss due to delay vanishes as search frictions vanish, that is, whether the equality $\lim_{\delta \to 1} \sup_{\sigma \in \Sigma(\delta)} L_\sigma(x, \delta) = 0$ holds for any player x where $\Sigma(\delta)$ denotes the set of all subgame perfect equilibria of the game with discount factor δ. Examples 4 and 5 show that equilibrium efficiency loss due to delay may remain as search frictions vanish.

Example 4 *(Wait and see).* The initial market is described by the following lists:

$$P(m_1) = w_1, w_2, \qquad P(w_1) = m_2, m_1,$$
$$P(m_2) = w_2, w_1, \qquad P(w_2) = m_1, m_2.$$

Each player receives a payoff of 3 from marrying the first choice and 1 from marrying the second choice. The contact function satisfies $C(m, w, S) = C(\hat{m}, \hat{w}, S)$ for any pairs $(m, w) \in S$ and $(\hat{m}, \hat{w}) \in S$ for any $S \in \mathscr{S}$.

Fix $\eta \in (0.5, 1)$. Given δ sufficiently close to 1 there exists $\tau(\delta) \in \mathbb{N}$ such that $0.5 < \delta^{\tau(\delta)}$ and $\delta^{\tau(\delta)-1} < \eta$. Consider the strategy profile $\sigma(\delta)$ specified by an automaton with the following states:

q_0: The initial state. In q_0, each man accepts no one; each woman accepts her first choice. The transition rules are

$$q_0 \longrightarrow \begin{cases} q_M & \text{if } (m_1, w_1) \text{ or } (m_2, w_2) \text{ meet on the } \tau(\delta)\text{th day,} \\ q_W & \text{if } (m_1, w_2) \text{ or } (m_2, w_1) \text{ meet on the } \tau(\delta)\text{th day,} \\ q_3 & \text{if some pair marry before the } \tau(\delta)\text{th day,} \\ q_0 & \text{otherwise.} \end{cases}$$

q_M: An absorbing state. When the state has just become q_M the remaining market is the initial one. In q_M the players follow the μ^M-strategy profile where μ^M is the men-optimal matching for the initial market.

q_W: Symmetric to q_M in which the μ^W-strategy profile is followed where μ^W is the women-optimal matching for the initial market.

q_3: An absorbing state. The remaining pair (m, w) accept each other.

Note that $\sigma(\delta)$ satisfies the private-dinner condition. Under $\sigma(\delta)$, no player marries during the first $\tau(\delta) - 1$ days. The pair that meet on the $\tau(\delta)$th day marry on that day and the other pair marry on the next day. It is easily verified that μ^M and μ^W both obtain with probability 0.5, and that on the $\tau(\delta) - n$th day player x's expected payoff $K(x, n, \delta)$ is greater than $\delta^{n+1}(0.5 \times 3 + 0.5 \times 1) = 2\delta^{n+1}$ for any n such that $0 < \tau(\delta) - n < \tau(\delta)$. By the choice of $\tau(\delta)$, $2\delta^{n+1} \geq 2\delta^{\tau(\delta)} > 1$. To verify that $\sigma(\delta)$ is a subgame perfect equilibrium given δ sufficiently close to 1, it suffices to check the absence of profitable one-deviations in q_0 because Proposition 4.2 and Lemma 4.1 cover the other states. Suppose (m_1, w_1) meet in q_0 on the $\tau(\delta) - n$th day. w_1's expected payoff from accepting m_1 is 1 whereas that from rejecting him is $K(w_1, n, \delta) > 1$, thus rejecting m_1 is optimal. m_1's expected payoffs from accepting and rejecting w_1 are the same as he will be rejected anyway, thus rejecting w_1 is optimal. The other cases are similar or can be verified by routine inspection.

Note that $V_{\sigma(\delta)}(x, \delta) < \delta^{\tau(\delta)-1}(0.5 \times 3 + 0.5 \times 1) = 2\delta^{\tau(\delta)-1}$ for any player x since x cannot marry before the $\tau(\delta)$th day. Thus

$$L_{\sigma(\delta)}(x,\delta) = 0.5 \times 3 + 0.5 \times 1 - V_{\sigma(\delta)}(x,\delta) > 2(1-\delta^{\tau(\delta)-1}) > 2(1-\eta) > 0.$$

That $\sigma(\delta)$ is a subgame perfect equilibrium of the game with discount factor δ sufficiently close to 1 implies $\lim_{\delta\to 1}\sup_{\sigma\in\Sigma(\delta)} L_\sigma(x,\delta) \geq 2(1-\eta)$ for any x. □

In Example 4, each player waits to see if oneself will be lucky to marry his or her first choice, the revelation of which is on the $\tau(\delta)$th day. As search frictions vanish, players become increasingly willing to wait longer. Efficiency loss lingers as the length of waiting grows in pace with the vanishing search frictions.

Example 5 *(War of attrition).* Take the game in Example 4. Consider strategy profile $\sigma(\delta)$ under which each player accepts his or her first choice in the remaining market with certainty and second choice (if there is one) with probability $q(\delta) \in (0, 1)$. Note that $\sigma(\delta)$ satisfies the Markov condition. We want to choose $q(\delta)$ so that $\sigma(\delta)$ is a subgame perfect equilibrium. Note that when the remaining market is the initial one, w_1 randomizes between accepting and rejecting m_1. Conditional on w_1 having rejected m_1 on a given day, all of the following four events occur with probability $q(\delta)/4$: (1) (m_1, w_1) marry tomorrow, (2) (m_2, w_1) marry tomorrow, (3) (m_2, w_2) marry tomorrow, and then (m_1, w_1) marry on the day after tomorrow, (4) (m_1, w_2) marry tomorrow, and then (m_2, w_1) marry on the day after tomorrow. With the remaining probability $1-q(\delta)$ no one marries tomorrow. To make w_1 indifferent between accepting and rejecting m_1, we must have

$$v(m_1,w_1) = \delta\Bigg[\frac{q(\delta)}{4}\Big(v(m_1,w_1) + v(m_2,w_1) + \delta v(m_1,w_1) + \delta v(m_2,w_1)\Big) + (1-q(\delta))v(m_1,w_1)\Bigg].$$

Substituting in $v(m_1, w_1) = 1$ and $v(m_2, w_1) = 3$ we conclude that $q(\delta) = (1-\delta)/\delta^2$. The same indifference argument applies to each of the other players. $(1-\delta)/\delta^2 \in (0, 1)$ for any $\delta > (\sqrt{5}-1)/2 \approx 0.618$. It is easily verified that for δ sufficiently close to 1, $\sigma(\delta)$ is a subgame perfect equilibrium if and only if $q(\delta) = (1-\delta)/\delta^2$.

Now evaluate the efficiency loss due to delay. $\sigma(\delta)$ induces a lottery in which each player marries either player on the other side with probability 0.5. For any player x and $\delta > (\sqrt{5}-1)/2$ obviously $V_{\sigma(\delta)}(x,\delta) = 1/\delta$. Thus

$$L_{\sigma(\delta)}(x,\delta) = 0.5 \times 3 + 0.5 \times 1 - 1/\delta.$$

Note that $L_{\sigma(\delta)}(x,\delta)$ is positive for any $\delta > (\sqrt{5}-1)/2$ and $\lim_{\delta\to 1} L_{\sigma(\delta)}(x,\delta) = 1$. Thus $\lim_{\delta\to 1}\sup_{\sigma\in\Sigma(\delta)} L_\sigma(x,\delta) \geq 1$. □

Example 5 resembles a war of attrition. A player "chickens out" to accept his or her second choice with probability $(1-\delta)/\delta^2$, which tends to 0 as δ tends to 1. It is notable that $L_{\sigma(\delta)}(x,\delta)$ increases with δ: Ironically, the expected total cost of search goes up as the average cost goes down. In both Examples 4 and 5, the efficiency gain from vanishing search frictions is undone in equilibrium by increasing delay in such a way that the incentives of the players are preserved, leaving the net efficiency loss significant.

4.7. A sufficient condition for equilibrium uniqueness

In this subsection a condition on the preference structure is identified under which the game has an essentially unique subgame perfect equilibrium if search frictions are small.

For $h \in Z$ define $o(h) := \big(\mu_h(m), t_h(m)\big)_{m\in M}$ where μ_h is the outcome matching of h and $t_h(m) \in \{1, 2, \ldots\}$ is the date of $(m, \mu_h(m))$'s marriage under h if $\mu_h(m) \in W$ or the date of the last day of h if $\mu_h(m) = s$. $t_h(m) = \infty$ if $\mu_h(m) = s$ and h is infinite. $o(h)$ records each realized marriage and its date under h. If $o(h) = o(h')$ then under h and h' every player is married to the same person on the same day or else stays single. Two strategy profiles are **outcome equivalent** if they induce the same probability measure on $\{o(h) : h \in Z\}$. A game has an **essentially unique subgame perfect equilibrium** if all of its subgame perfect equilibria are outcome equivalent.

A pair (m, w) is **woman-acceptable** if m is acceptable to w. Lemma 4.1(b) implies equilibrium marriages may occur only between woman-acceptable pairs. For a submarket S, player $x \in S$ is a **top player** for S if x and some $y \in S$ form a top pair for S and moreover one of the following is true: (1) $|A^S(x)| = 1$ or (2) $|A^S(x)| > 1$ and x is a top player for $S\backslash(m, w)$ for any woman-acceptable pair $(m, w) \in S$ such that $x \notin \{m, w\}$.

An immediate observation is that if x is a top player for S, and S' is derived from S as a result of a sequence of other woman-acceptable pairs having left, then either $|A^S(x)| = 0$ or x is still a top player for S'. The observation also implies that, despite the recursive definition, whether x is a top player for S can be verified mechanically in a finite number of steps. Since x (assumedly male) being a top player for S implies he is part of a top pair for S, it follows that he continues to be part of a top pair for the remaining market as other woman-acceptable pairs leave S (although his partner in the current top pair might change), until x can no longer find an acceptable woman who also finds him acceptable. In the case that every man is acceptable to every woman and vice versa, x being a top player for the initial market implies he is the favorite man of his favorite woman in the remaining market at any moment when he is still unmarried.

Proposition 4.8. *Suppose there is an ordering of the men $m_1, \ldots, m_{|M|}$, an ordering of the women $w_1, \ldots, w_{|W|}$ and a positive integer k such that:*

1. *For any $i \le k$, (m_i, w_i) is a top pair for $S_i := (\{m_j : j \ge i\}, \{w_j : j \ge i\}, u, v)$, and moreover m_i or w_i is a top player for S_i.*
2. *$S_{k+1} := (\{m_j : j \ge k+1\}, \{w_j : j \ge k+1\}, u, v)$ is trivial.*

There exists $d < 1$ such that for any $\delta > d$, (M, W, u, v, C, δ) has an essentially unique subgame perfect equilibrium.

The proof is provided in Appendix A.2. Given the premise of Proposition 4.8, the initial market has a unique stable matching μ such that $\mu(m_i) = w_i$ for $i \le k$, and $\mu(m_i) = \mu(w_i) = s$ for $i > k$. To sketch out the proof, for simplicity suppose every man is acceptable to every woman and vice versa. The first step is proving by induction on the market size that a top player w_1 (assumedly female) for the initial market marries her favorite man almost surely and upon first meeting in any subgame perfect equilibrium for δ sufficiently close to 1. The inductive step goes as follows. Suppose m_1 is w_1's favorite man in the initial market. By Lemma 4.5, (m_1, w_1) accept each other when they meet. Fix an equilibrium. Suppose w_1 deviates to an alternative strategy under which she accepts only m_1 when the remaining market is the initial market, and switches back to following the equilibrium strategy after someone has married. For δ close to 1, w_1's expected payoff from the deviation is strictly larger than $v(m', w_1)$ where m' is w_1's second favorite man, because if someone has married before (m_1, w_1)'s first meeting then in the continuation subgame, as the market has become smaller to which the inductive hypothesis is applicable, w_1 marries her favorite man in the remaining market, which can be m_1 or m', almost surely and upon first meeting, whereas if (m_1, w_1) meet before anyone has married then they marry each other. Thus w_1's equilibrium payoff must as well be strictly larger than $v(m', w_1)$, implying if no one has married, w_1 rejects any man worse than m_1. Given w_1's behavior pinned down, it is shown that m_1's best response is accepting only w_1 when no one has married. Combined with the inductive hypothesis, it follows that (m_1, w_1) marry almost surely and upon first meeting. With (m_1, w_1)'s equilibrium behavior determined this way, we may treat them as nonstrategic "dummy players" who only accept each other. Then since one of m_2 or w_2 is a top player for the submarket without (m_1, w_1), applying the analogous logic it is shown that (m_2, w_2) also marry almost surely and upon first meeting, and so on and so forth until the market unravels to the trivial S_{k+1}. From Lemma 4.1 it follows that every player in S_{k+1} must stay single.

The condition proposed in Proposition 4.8, for convenience referred to as Condition 4.8, is stronger than the Sequential Preference Condition. The initial market from Example 3, for instance, satisfies the Sequential Preference Condition but fails Condition 4.8. Indeed, in Example 3, although (m_1, w_1) is a top pair for the initial market, if w_1 goes "astray" and marries m_3 then in the remaining market m_1 is no longer part of a top pair, and consequently his search prospect drops drastically because he has no chance of marrying his second choice w_2. It is this possibility of a large drop in the search prospect that makes m_1 willing to accept his second choice, w_2, even when w_1 is still in the market. For a market that satisfies Condition 4.8, in contrast, if m (assumedly male) is a top player for the initial market and forms a top pair with w, then even if w marries someone else, in the remaining market m still forms a top pair with w' whom he likes just next to w. Loosely speaking, the search prospect of m would only experience a gradual drop if w marries someone else. This ensures that m will wait for w if she is still in the market, preventing self-confirmation of mutual doubt present in Example 3 from arising.

An acyclicity condition under which a marriage market has a unique stable matching is proposed in Romero-Medina and Triossi (2013),[11] which requires that the preferences of the players on one side over acceptable players on the other side do not display cycles. It can be shown that the acyclicity condition implies Condition 4.8. Condition 4.8 is weaker, most importantly because it allows preference cycles. For instance, the market with preference lists

$$P(m_1) = w_1, w_2, \qquad P(w_1) = m_1, m_2,$$
$$P(m_2) = w_2, w_1, \qquad P(w_2) = m_2, m_1$$

displays preference cycles on both sides but satisfies Condition 4.8.[12] It is worthwhile noting that Condition 4.8 is an implication of the commonly seen preference structure such that players on at least one side of the market share the same preference ordering over those on the other side.

[11] I am indebted to a referee for referring me to this paper.

[12] A weaker condition, the absence of simultaneous cycles, is proposed in Romero-Medina and Triossi (2013) that also guarantees the stable matching is unique. Condition 4.8 is neither necessary nor sufficient for that weaker condition. The initial market from Example 3 satisfies the absence of simultaneous cycles but violates Condition 4.8. An example that violates the absence of simultaneous cycles but satisfies Condition 4.8 is given in Appendix A.4.

5. Conclusion

This paper studies a search and matching game with a marriage market embedded, and analyzes whether matchings that arise in equilibria are stable when search frictions are small. It is found that this is not the case in general. Unstable matchings may arise for many reasons and under restrictive conditions. Moreover, significant loss of efficiency due to delay may be incurred in equilibria even if search frictions are small. A condition that implies preference alignment in a strong sense ensures equilibrium uniqueness, restoring stability and efficiency.

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Appendix A

A.1. Proof of Proposition 4.7

Proposition 4.7 is proved with the assistance of Lemma A.1.

Lemma A.1. *For any game (M, W, u, v, C, δ) there exist $\tau \in \mathbb{N}^+$, $\overline{q} \in (0, 1)$ and $(m, w) \in M \times W$ such that for any subgame perfect equilibrium the following is true: If a meeting between (m, w) ends in separation with positive probability, then the probability that no player marries during the first τ days of the subgame resulting from that meeting having ended in separation is less than $\overline{q}$.*

Proof. Let S_I denote the initial market. There exists a pair (m, w) such that $m = \alpha^{S_I}(w)$ because S_I is nontrivial. Let $\overline{w}$ and $\underline{w}$ respectively denote m's first and last choices in $A^{S_I}(m)$ and define $\overline{\pi} := u(m, \overline{w})$ and $\underline{\pi} := u(m, \underline{w})$. Thus $\overline{\pi} \geq u(m, w) \geq \underline{\pi} > 0$ and therefore we can pick $\tau \in \mathbb{N}^+$ and $\overline{q} \in (0, 1)$ such that $\delta[(1 - \overline{q})\overline{\pi} + \overline{q}\delta^\tau\overline{\pi}] < \underline{\pi}$.

Fix a subgame perfect equilibrium σ. Suppose (m, w) meet on a day. Lemma 4.1(d) implies w accepts m if she has been accepted since $m = \alpha^{S_I}(w)$. Thus if the meeting ends in separation with positive probability then it must be that m rejects w with positive probability. Let q denote the equilibrium probability that no player marries during the first τ days in the subgame resulting from m having rejected w. By Lemma 4.1(d), m's expected payoff from rejecting w is no more than $\delta[(1 - q)\overline{\pi} + q\delta^\tau\overline{\pi}]$. If m rejects w with positive probability in equilibrium then $\delta[(1 - q)\overline{\pi} + q\delta^\tau\overline{\pi}] \geq u(m, w) \geq \underline{\pi}$ because m's expected payoff from accepting w is $u(m, w)$ as he will be accepted. Thus $q < \overline{q}$ by the choice of $\overline{q}$. □

Proof of Proposition 4.7. Fix a game (M, W, u, v, C, δ) and denote the initial market as S_I. Prove by induction. Suppose $|M| = 1$. Let m be the only man and $w := \alpha^{S_I}(m)$. Thus (m, w) is a top pair for S_I. In any subgame perfect equilibrium the game ends with certainty after (m, w) meet because they marry each other upon first meeting by Lemma 4.5. The proposition then follows.

Suppose the proposition is true if $|M| < n$ for some n. Consider $|M| = n$. Fix a subgame perfect equilibrium σ. By Lemma A.1 there exist $\tau \in \mathbb{N}^+$, $\overline{q} \in (0, 1)$ and $(m, w) \in S_I$ such that if (m, w) meet and in equilibrium the meeting ends in separation with positive probability, then the probability that no one marries during the first τ days of the subgame resulting from that meeting having ended in separation is less than $\overline{q}$. For any $h \in \hat{H}$ such that $S(h) = S_I$ let $\beta(h)$ denote the equilibrium probability that no one marries in the first $\tau + 1$ days of $\Gamma(h)$; let $\phi(h)$ denote the equilibrium probability that (m, w) marry on the first day of $\Gamma(h)$ conditional on them meeting on that day. Then by Lemma A.1, for any $h \in \hat{H}$

$$\beta(h) \leq \epsilon\Big(\phi(h) \times 0 + (1 - \phi(h)) \times \overline{q}\Big) + (1 - \epsilon) \leq \epsilon\overline{q} + (1 - \epsilon). \tag{I1}$$

Let h_0 denote the initial (empty) history. For any $k \in \mathbb{N}^+$ let $\hat{H}_k$ denote the set of all histories h in $\hat{H}$ such that the first day of $\Gamma(h)$ is the $k(\tau + 1) + 1$st day of the initial game and $S(h) = S_I$. Thus $h \in \hat{H}_k$ being reached implies no player has married during the first $k(\tau + 1)$ days of the initial game. Let E_k denote the event that any $h \in \hat{H}_k$ is reached. Inequality I1 implies $\Pr_\sigma(E_1) \leq \epsilon\bar{q} + (1 - \epsilon)$ because $h_0 \in \hat{H}$. I1 also implies $\Pr_\sigma(E_k|E_{k-1}) \leq \epsilon\bar{q} + (1 - \epsilon)$ for any $k > 1$. By the inductive hypothesis the probability Q that the game does not end at all is equal to the probability that every player remains in the market into the infinite future. Hence for any $k > 1$ we have $Q \leq \Pr_\sigma(E_k) = \Pr_\sigma(E_1) \prod_{j=2}^{k} \Pr_\sigma(E_j|E_{j-1})$. The present proposition follows as $Q \leq \lim_{k\to\infty}[\epsilon\overline{q} + (1 - \epsilon)]^k = 0$. □

A.2. *Proof of Proposition 4.8*

Introduce additional terminology for the proof. Let $\Sigma(\delta)$ denote the set of all subgame perfect equilibria of the game with discount factor δ in the environment $E := (M, W, u, v, C)$. A (possibly empty) set P of disjoint pairs of a man and a woman from the initial market $S_I := (M, W, u, v)$ is a **settled set** for E if there exists some $d < 1$ such for any $\sigma \in \Sigma(\delta)$ where $\delta > d$, σ implies any $(m, w) \in P$ marry almost surely and upon first meeting in $\Gamma(h)$ if $h \in \hat{H}$, $(m, w) \in S(h)$, and moreover $S(h)$ can be derived from S_I as a result of a (possibly empty) sequence of pairs having left, among which each pair (m', w') is either in P or is a woman-acceptable pair from $S_I\backslash P$, where $S_I\backslash P$ denotes the submarket that complements P in S_I. d is called a **settling discount factor** for P. Note that any $d' > d$ is also a settling discount factor for P.

Proposition 4.8 is proved with the assistance of Lemma A.2.

Lemma A.2. *For any settled set P for the environment E, if m is a top player for $S_I\backslash P$ and $w := \alpha^{S_I\backslash P}(m)$, or w is a top player for $S_I\backslash P$ and $m := \alpha^{S_I\backslash P}(w)$, then $P \cup \{(m, w)\}$ is also a settled set for E.*

Proof. Prove by induction on $|M|$. Suppose $|M| = 1$. Let m be the only man and $w := \alpha^{S_I}(m)$. For δ sufficiently close to 1, it is easy to see that in any subgame perfect equilibrium, (m, w) marry almost surely and upon first meeting. Thus there are two settled sets, the empty set P_1 and $P_2 := \{(m, w)\}$. The lemma is true for P_1 because m is a top player in $S_I = S_I\backslash P_1$ and as is concluded $\{(m, w)\}$ is a settled set. The lemma is true for P_2 vacuously.

Suppose the lemma is true if $|M| < n$ for some n. Consider $|M| = n$. Pick a settled set P for E and let $d < 1$ be a settling discount factor for P. Suppose $\overline{w}$ is a top player for $\overline{S} := S_I\backslash P$. Let $\overline{m} := \alpha^{\overline{S}}(\overline{w})$ and denote $\underline{m}$ as $\overline{w}$'s second choice in $A^{\overline{S}}(\overline{w}) \cup \{s\}$. By definition of the top player we have $\overline{w} = \alpha^{\overline{S}}(\overline{m})$. Notice that $\underline{m}$ may be s. Define $\hat{H}_I := \{h \in \hat{H} : S(h) = S_I\}$. Thus $\Gamma(h)$ is isomorphic to the initial game for $h \in \hat{H}_I$. Consider $\sigma \in \Sigma(\delta)$ for $\delta > d$. It is helpful to bear in mind that the choice of σ depends on δ. In σ, by assumption $\overline{w}$ may only marry a man from $\overline{S}$ in the subgame $\Gamma(h)$ where $h \in \hat{H}_I$, thus her expected payoff in $\Gamma(h)$ is no higher than $v(\overline{m}, \overline{w})$ by an argument similar to the proof for Lemma 4.1(d). It follows that $\overline{w}$ always accepts $\overline{m}$ when the remaining market is S_I. Given that, $\overline{m}$ always accepts $\overline{w}$ when the remaining market is S_I by a similar argument. Thus in σ, (m, w) marry upon first meeting in $\Gamma(h)$ for any $h \in \hat{H}_I$.

Suppose $\overline{w}$ deviates to the strategy under which she only accepts $\overline{m}$ when the remaining market is S_I, and switches back to following σ if the remaining market is no longer S_I. Let σ' denote the strategy profile due to $\overline{w}$'s unilateral deviation. For each $h \in \hat{H}_I$ let $K(h, \delta, \sigma)$ denote $\overline{w}$'s expected payoff in $\Gamma(h)$ under σ'. Define $\underline{K}(\delta, \sigma) := \inf_{h \in \hat{H}_I} K(h, \delta, \sigma)$. Let $p(h, \sigma, m, w)$ denote the probability that $(m, w) \in S_I$ meet and marry on the first day of $\Gamma(h)$ under σ'. Let $h'(h, m, w) \in \hat{H}$ denote the immediate history resulting from (m, w) having married on the first day of $\Gamma(h)$. For $w \neq \overline{w}$ let $U(h, \delta, \sigma, m, w)$ denote $\overline{w}$'s expected payoff in $\Gamma(h'(h, m, w))$ under σ'. σ instead of σ' appears as an argument for K, p and U because σ' is determined by σ. By the definition of $\underline{K}$, for any $\eta_1 > 0$ there exists $h_{\eta_1} \in \hat{H}_I$ such that

$$\begin{aligned}
\underline{K}(\delta, \sigma) + \eta_1 > K(h_{\eta_1}, \delta, \sigma) \\
&\geq p(h_{\eta_1}, \sigma, \overline{m}, \overline{w}) v(\overline{m}, \overline{w}) \\
&\quad + \sum_{m \in M} \sum_{w \neq \overline{w}} p(h_{\eta_1}, \sigma, m, w) \delta U(h_{\eta_1}, \delta, \sigma, m, w) \\
&\quad + \Big(1 - p(h_{\eta_1}, \sigma, \overline{m}, \overline{w}) - \sum_{m \in M} \sum_{w \neq \overline{w}} p(h_{\eta_1}, \sigma, m, w)\Big) \delta \underline{K}(\delta, \sigma). \qquad \text{(I2)}
\end{aligned}$$

$p(h_{\eta_1}, \sigma, \overline{m}, \overline{w}) > \epsilon$ because $(\overline{m}, \overline{w})$ accept each other in $\Gamma(h_{\eta_1})$. Fix (m, w) such that $w \neq \overline{w}$ and $p(h_{\eta_1}, \sigma, m, w) > 0$. Because players other than $\overline{w}$ follow σ, $(m, w) \in P$ or (m, w) is a woman-acceptable pair from $\overline{S}$. Let $P' := P \backslash \{(m, w)\}$ if $(m, w) \in P$ or $P' := P$ otherwise. Let $\overline{S}'$ denote the submarket complementing P' in $S(h'(h, m, w)) = (M \backslash \{m\}, W \backslash \{w\}, u, v)$. If $(m, w) \in P$ then $\overline{S}' = \overline{S}$ and thus $\overline{w}$ is a top player for $\overline{S}'$ and $\alpha^{\overline{S}'}(\overline{w}) = \overline{m}$. If instead (m, w) is a woman-acceptable pair from $\overline{S}$ then $\overline{S}' = \overline{S} \backslash (m, w)$ and one of the following is true:

1. $m \neq \overline{m}$: $\overline{w}$ is a top player for $\overline{S}'$ and $\alpha^{\overline{S}'}(\overline{w}) = \overline{m}$.
2. $m = \overline{m}$, $\underline{m} \neq s$: $\overline{w}$ is a top player for $\overline{S}'$ and $\alpha^{\overline{S}'}(\overline{w}) = \underline{m}$.
3. $m = \overline{m}$, $\underline{m} = s$: $\alpha^{\overline{S}'}(\overline{w}) = s$.

It is easy to verify from definition that P being a settled set for E implies P' is a settled set for the sub-environment $E' := (M \backslash \{m\}, W \backslash \{w\}, u, v, C)$. Note that σ' restricted to $\Gamma(h'(h_{\eta_1}, m, w))$ is a subgame perfect equilibrium of it because $\overline{w}$ has switched back to following σ. If $(m, w) \in P$, or if case 1 above is true, then since the number of men in $S(h'(h, m, w))$ is less than n, the inductive hypothesis applies, implying $P' \cup \{(\overline{m}, \overline{w})\}$ is a settled set for E', thus for δ sufficiently close to 1, $\sigma \in \Sigma(\delta)$ implies $(\overline{m}, \overline{w})$ marry almost surely and upon first meeting in $\Gamma(h'(h_{\eta_1}, m, w))$, and thus $\delta U(h_{\eta_1}, \delta, \sigma, m, w) \to v(\overline{m}, \overline{w})$ as $\delta \to 1$. Likewise if case 2 is true then $\delta U(h_{\eta_1}, \delta, \sigma, m, w) \to v(\underline{m}, \overline{w})$ as $\delta \to 1$. If case 3 is true then by the assumption that P' is a settled set for E' and Lemma 4.1(a), $\delta U(h_{\eta_1}, \delta, \sigma, m, w) = 0 = v(\underline{m}, \overline{w})$. In general, for any $\eta_2 > 0$ there exists $d_{\eta_2} < 1$ such that if $\delta > d_{\eta_2}$ then for $w \neq \overline{w}$, $p(h_{\eta_1}, \sigma, m, w) > 0$ implies

$$v(\overline{m},\overline{w}) > \delta U(h_{\eta_1},\delta,\sigma,m,w) > v(\underline{m},\overline{w}) - \eta_2. \tag{I3}$$

Substituting I3 along with $p(h_{\eta_1},\sigma,\overline{m},\overline{w}) > \epsilon$, $\underline{K}(\delta,\sigma) \le v(\overline{m},\overline{w})$ and $R(h_{\eta_1},\sigma) := \sum_{m\in M}\sum_{w\neq\overline{w}} p(h_{\eta_1},\sigma,m,w)$ back to I2, we have

$$\underline{K}(\delta,\sigma) > \frac{\epsilon v(\overline{m},\overline{w}) + R(h_{\eta_1},\sigma)(v(\underline{m},\overline{w}) - \eta_2) - \eta_1}{1-\delta(1-\epsilon-R(h_{\eta_1},\sigma))} \tag{I4}$$

for η_2 sufficiently small and $\delta > d_{\eta_2}$. The right hand side of I4 is decreasing in $R(h_{\eta_1},\sigma)$ for δ close 1 and η_1,η_2 close to 0. For such extreme values, $R(h_{\eta_1},\sigma) < 1-\epsilon$ implies for $\delta > d_{\eta_2}$,

$$\underline{K}(\delta,\sigma) > \epsilon v(\overline{m},\overline{w}) + (1-\epsilon)v(\underline{m},\overline{w}) - (1-\epsilon)\eta_2 - \eta_1.$$

Taking limits $\eta_1 \to 0$, $\eta_2 \to 0$ and correspondingly $\delta \to 1$ we have

$$\lim_{\delta\to 1}\inf_{\sigma\in\Sigma(\delta)} \underline{K}(\delta,\sigma) \ge \epsilon v(\overline{m},\overline{w}) + (1-\epsilon)v(\underline{m},\overline{w}) > v(\underline{m},\overline{w}). \tag{I5}$$

Inequality I5 implies $\overline{w}$'s expected payoff under σ in $\Gamma(h)$ where $h \in \hat{H}_I$ is strictly greater than $v(\underline{m},\overline{w})$ for δ sufficiently close to 1. For such δ, $\sigma \in \Sigma(\delta)$ implies $\overline{w}$ rejects any m such that $\overline{m} \succ_{\overline{w}} m$ when the remaining market is S_I.

Now consider $\overline{m}$. Suppose other players (including $\overline{w}$) follow σ and $\overline{m}$ deviates to the strategy under which he only accepts $\overline{w}$ when the remaining market is S_I, and switches back to following σ if the remaining market is no longer S_I. Let σ'' denote the strategy profile due to $\overline{m}$'s unilateral deviation.

Claim A.3. *For δ sufficiently close to 1, $\sigma \in \Sigma(\delta)$ implies under σ'', $(\overline{m},\overline{w})$ marry almost surely and upon first meeting in $\Gamma(h)$ for any $h \in \hat{H}_I$.*

Proof. Observe that for δ sufficiently close to 1, almost surely one of the following scenarios occur under σ'':

1. $(\overline{m},\overline{w})$ meet when the remaining market is S_I.
2. The first marriage is between (m,w) where $m \neq \overline{m}$, $w \neq \overline{w}$, $(m,w) \in P$ or (m,w) is a woman-acceptable pair from $\overline{S}$.

That $\overline{w}$ will not marry with $m \neq \overline{m}$ as the first married couple is due to the assumption that she may not marry a man from a pair in P, plus the conclusion from the above that she rejects every man worse than $\overline{m}$ when the remaining market is S_I. If scenario 1 occurs then $(\overline{m},\overline{w})$ marry immediately. If scenario 2 occurs, then the inductive hypothesis[13] implies $(\overline{m},\overline{w})$ marry almost surely and upon first meeting in the subsequent subgame. □

By Claim A.3, $\overline{m}$'s expected payoff under σ'' in $\Gamma(h)$ where $h \in \hat{H}_I$ tends to $u(\overline{m},\overline{w})$ as $\delta \to 1$, implying $\overline{m}$'s expected payoff under σ in $\Gamma(h)$ also tends to $u(\overline{m},\overline{w})$. Thus for δ sufficiently close to 1, $\sigma \in \Sigma(\delta)$ implies when the remaining market is S_I, $\overline{m}$ rejects any w such that $\overline{w} \succ_{\overline{m}} w$ if w would accept $\overline{m}$ with positive probability. Then an analogous claim to Claim A.3 is true for σ with an analogous proof.

[13] The argument for the applicability of the inductive hypothesis, and its actual application, are the same as those used above for $\overline{w}$'s case.

Now consider any $h \in \hat{H}$ such that $S(h) \neq S_I$, $(\overline{m}, \overline{w}) \in S(h)$, and moreover $S(h)$ can be derived from S_I as a result of a sequence of pairs having left, among which each pair (m, w) is either in P or is a woman-acceptable pair from $\overline{S}$. Let $P(h)$ denote the set of pairs in P which are present in $S(h)$, and let $E(h)$ denote the sub-environment with initial market $S(h)$. It is easy to verify from definition that $P(h)$ is a settled set for $E(h)$, $\overline{w}$ is a top player for $\overline{S}(h) := S(h)\backslash P(h)$, and $\overline{m} = \alpha^{\overline{S}(h)}(\overline{w})$. Thus the inductive hypothesis implies $P(h) \cup \{(\overline{m}, \overline{w})\}$ is a settled set for $E(h)$. Therefore for δ sufficiently close to 1, $\sigma \in \Sigma(\delta)$ implies $(\overline{m}, \overline{w})$ marry almost surely and upon first meeting in $\Gamma(h)$. It follows that $P \cup \{(\overline{m}, \overline{w})\}$ is a settled set for E.

The proof for the case that a male player is a top player for $\overline{S}$ is similar. □

Proof of Proposition 4.8. Denote $E := (M, W, u, v, C)$. To prove the present proposition it suffices to show there exists some $d < 1$ such that for any $\sigma \in \Sigma(\delta)$ where $\delta > d$, (m_i, w_i) marry almost surely and upon first meeting if $i \leq k$, and m_i or w_i stays single if $i > k$.

Since the empty set is a settled set for E and m_1 or w_1 is a top player for the initial market S_1, Lemma A.2 implies $P_1 := (\{m_1, w_1\})$ is a settled set for E. Let $P_i := \{(m_j, w_j) : j \leq i\}$. Suppose P_i is a settled set for E if $i < n$ for some $n \leq k$. Since m_n or w_n is a top player for $S_n = S_1\backslash P_{n-1}$, Lemma A.2 implies that P_n is a settled set for E. Thus P_k is a settled set for E. Let d be a settling discount factor for P_k. Pick $\sigma \in \Sigma(\delta)$ where $\delta > d$. P_k being a settled set for E implies (m_i, w_i) marry almost surely and upon first meeting under σ if $i \leq k$. Pick a man $m \in S_{k+1}$. (m, w) marry with positive probability in σ only if $w \in S_{k+1}$ and m is acceptable to w. Since S_{k+1} is trivial, if m is acceptable to $w \in S_{k+1}$ then w is unacceptable to m. Thus if m marries at all with positive probability, his expected payoff would be negative. Lemma 4.1(a) then implies m stays single in σ. It follows that a woman in S_{k+1} stays single in σ as well. □

A.3. *Example: a man might marry an unacceptable woman*

The following example demonstrates that a man may marry an unacceptable woman with positive probability in equilibrium.

The initial market is described by the following lists:

$$P(m_1) = w_1, w_2, \qquad P(w_1) = m_2, m_1,$$
$$P(m_2) = w_2, w_1, \qquad P(w_2) = m_1, m_2,$$
$$P(m_3) = w_3, \qquad P(w_3) = m_3, m_1.$$

Each of the players m_1, m_2, w_1, w_2 receives a payoff of 16 from marrying the first choice, 1 the second choice, and -1 the unacceptable choice; m_3 receives a payoff of 16 from marrying w_3; w_3 receives a payoff of 16 from marrying m_3 and 4 from marrying m_1. Set $\delta = 0.5$. The contact function satisfies $C(m, w, S) = C(\hat{m}, \hat{w}, S)$ for any pairs $(m, w) \in S$ and $(\hat{m}, \hat{w}) \in S$ for any $S \in \mathscr{S}$.

Consider strategy profile σ given as follows:

- If (m_1, w_3) do not meet on the first day, the players follow the μ^m strategy profile where μ^m is the men-optimal matching for the initial market.
- If (m_1, w_3) meet on the first day, m_1 accepts w_3 with certainty and w_3 accepts m_1 with probability 0.5.

– If m_1 accepts w_3 on the first day, then regardless of whether m_1 is accepted, after the first day the players follow the $\hat{\mu}^m$-strategy profile where $\hat{\mu}^m$ is the men-optimal matching for the remaining market.
– If m_1 rejects w_3 on the first day, then after the first day the players follow the μ^w strategy profile where μ^w is the women-optimal matching for the initial market.

It is straightforward to verify that σ is a subgame perfect equilibrium of the game. Notably, if (m_1, w_3) meet on the first day, m_1 accepts w_3 with positive probability despite the latter being unacceptable, because by doing so there is a probability of 0.5 that w_3 will reject m_1 and in the subsequent subgame m_1 will marry his first choice w_1 eventually, whereas if he rejects w_3 he will have to marry his second choice w_2 eventually. Meanwhile, because of discounting, w_3 is indifferent between marrying m_1 today and marrying her first choice m_3 in the future, justifying her randomizing between accepting and rejecting m_1. Under σ there is a positive probability of $1/9 \times 0.5 = 1/18$ that m_1 marries the unacceptable w_3.

A.4. Example: a market that satisfies Condition 4.8 but violates the absence of simultaneous cycles

It is easy to verify that the marriage market described by the following lists of preferences satisfies Condition 4.8 but violates the absence of simultaneous cycles given in Romero-Medina and Triossi (2013).

$$\begin{aligned}
P(m_1) &= w_1, w_3, w_4, & P(w_1) &= m_1, \\
P(m_2) &= w_2, w_4, w_3, & P(w_2) &= m_2, \\
P(m_3) &= w_3, & P(w_3) &= m_3, m_2, m_1, \\
P(m_4) &= w_4, & P(w_4) &= m_4, m_1, m_2.
\end{aligned}$$

Observe that there are two preference cycles: $w_3 \succ_{m_1} w_4 \succ_{m_2} w_3$ and $m_2 \succ_{w_3} m_1 \succ_{w_4} m_2$, implying the simultaneous cycle $(w_4, m_1, w_3, m_2, w_4)$ in the notation of Romero-Medina and Triossi (2013).

References

Adachi, Hiroyuki, 2003. A search model of two-sided matching under nontransferable utility. J. Econ. Theory 113, 182–198.
Alcalde, José, 1996. Implementation of stable solutions to marriage problems. J. Econ. Theory 69, 240–254.
Alcalde, Jose, Romero-Medina, Antonio, 2000. Simple mechanisms to implement the core of college admissions problems. Games Econ. Behav. 31, 294–302.
Alcalde, José, Romero-Medina, Antonio, 2005. Sequential decisions in the college admissions problem. Econ. Lett. 86, 153–158.
Bloch, Francis, Diamantoudi, Effrosyni, 2011. Noncooperative formation of coalitions in hedonic games. Int. J. Game Theory 40, 263–280.
Bloch, Francis, Ryder, Harl, 2000. Two-sided search, marriages, and matchmakers. Int. Econ. Rev. 41, 93–115.
Burdett, Ken, Coles, Melvyn G., 1997. Marriage and class. Q. J. Econ. 112, 141–168.
Diamantoudi, Effrosyni, Miyagawa, Eiichi, Xue, Licun, 2015. Decentralized matching: the role of commitment. Games Econ. Behav. 92, 1–17.
Eeckhout, Jan, 1999. Bilateral search and vertical heterogeneity. Int. Econ. Rev. 40, 869–887.
Eeckhout, Jan, 2000. On the uniqueness of stable marriage matchings. Econ. Lett. 69, 1–8.
Gale, David, 2000. Strategic Foundations of General Equilibrium: Dynamic Matching and Bargaining Games. Cambridge University Press, Cambridge.

Gale, Douglas, Sabourian, Hamid, 2006. Markov equilibria in dynamic matching and bargaining games. Games Econ. Behav. 54, 336–352.

Gale, David, Shapley, Lloyd, 1962. College admissions and the stability of marriage. Am. Math. Mon. 69, 9–15.

Haeringer, Guillaume, Wooders, Myrna, 2011. Decentralized job matching. Int. J. Game Theory 40, 1–28.

Lauermann, Stephan, Nöldeke, Georg, 2014. Stable marriages and search frictions. J. Econ. Theory 151, 163–195.

McNamara, J.M., Collins, E.J., 1990. The job search problem as an employer-candidate game. J. Appl. Probab. 27, 815–827.

Niederle, Muriel, Yariv, Leeat, 2009. Decentralized matching with aligned preferences. NBER Working Papers 14840. National Bureau of Economic Research, Inc.

Osborne, Martin J., Rubinstein, Ariel, 1990. Bargaining and Markets. Academic Press, Inc., San Diego, CA.

Pais, Joana, 2008. Incentives in decentralized random matching markets. Games Econ. Behav. 64, 632–649.

Romero-Medina, Antonio, Triossi, Matteo, 2013. Acyclicity and singleton cores in matching markets. Econ. Lett. 118, 237–239.

Roth, Alvin E., 2008. What have we learned from market design? Econ. J. 118, 285–310.

Roth, Alvin E., Liveria Sotomayor, Marilda A., 1990. Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis. Cambridge University Press, Cambridge.

Roth, Alvin E., Vande Vate, John H., 1990. Random paths to stability in two-sided matching. Econometrica 58, 1475–1480.

Rubinstein, Ariel, Wolinsky, Asher, 1990. Decentralized trading, strategic behaviour and the Walrasian outcome. Rev. Econ. Stud. 57, 63–78.

Smith, Lones, 2006. The marriage model with search frictions. J. Polit. Econ. 114, 1124–1144.

Suh, Sang-Chul, Wen, Quan, 2008. Subgame perfect implementation of stable matchings in marriage problems. Soc. Choice Welf. 31, 163–174.