The Limits of Ex Post Implementation without Transfers *

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Abstract

We study ex post implementation in collective decision problems where monetary transfers cannot be used. We find that deterministic ex post implementation is impossible if the underlying environment is neither almost an environment with private values nor almost one with common values. Thus, desirable properties of ex post implementation such as informational robustness become difficult to achieve when preference interdependence and preference heterogeneity are both present in the environment.

JEL Codes: D82, D71

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Collective decision-making takes place everywhere, from a committee choosing which job candidates to hire, a congress deciding whether to pass a bill, to a country electing its next president. When designing a decision mechanism for such situations, an important consideration is informational robustness: The mechanism should function effectively for a wide range of information structures, i.e., what agents know and believe about each other's information. Robustness is important because decision mechanisms are often institutionalized for repeated use, each time tackling a new problem with a different information structure. Thus, robust, all-purpose mechanisms are best suited for institutions such as committees, legislatures or elections. Moreover, even in a single decision problem, there is usually uncertainty about the underlying information structure. Thus, narrowly tailored mechanisms may misfire if the actual information structure turns out to be different from what was expected.

One might then ask: Are robust decision mechanisms viable? If monetary transfers are allowed, then the answer can be positive — even if one requires robustness against *all* possible information structures, which, by Bergemann and Morris (2005), amounts to the mechanism in question admitting an *ex post equilibrium*. More specifically, it is known that in interdependent value environments, non-trivial, even efficient, social choice functions can be *ex post incentive compatible (EPIC)*, i.e., implementable in an ex post equilibrium of some mechanism, if private information is one-dimensional.[1] There are limits to ex post implementation with transfers, though, as Jehiel et al. (2006) show: If private information is continuous and multi-dimensional, then deterministic EPIC social choice functions must be constant in generic environments.

However, in many collective decision problems, including the examples mentioned above, monetary transfers cannot be used. Is non-trivial ex post implementation still possible *without transfers*? In this paper, we give a largely negative answer to this question. Specifically, for collective decision problems with a continuous state space, we show that if transfers are not allowed, then deterministic EPIC social choice functions must be constant as long as there is a "small amount" of preference interdependence and preference heterogeneity in the environment. If there are only two alternatives, then the conclusion even extends to stochastic social choice functions.

Our result is strong because it rules out the possibility of non-trivial ex post implementation in a broad range of environments. In particular, the sufficient condition, that there is a "small amount" of preference interdependence and heterogeneity, is easily satisfied regardless of whether information is one- or multi-dimensional, even in certain "non-generic"

[1] For public goods provision, see Section 5 in Chung and Ely (2003). In auction settings, efficient social choice functions are ex post implementable when preferences satisfy appropriate single-crossing conditions; see Bergemann and Välimäki (2002), Crémer and McLean (1985), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), Maskin (1992), and Perry and Reny (2002).

but economically significant exceptions to Jehiel et al. (2006) such as environments with separable preferences.

To elaborate on this point, let us describe the setting. A group of n agents must collectively choose one of finitely many alternatives. Each agent i's private information — her type — is a number or vector θ_i, whereas the collection of everyone's types, $\theta = (\theta_1, ..., \theta_n)$, constitutes the payoff-relevant *state*. An agent's preference over the alternatives depends on the state, which includes others' as well as her own information.

The precise sufficient condition for our theorem is as follows: if in state θ some agents are indifferent between two alternatives (a, b), then among the indifferent agents there exists a certain agent i whose indifference between (a, b) is broken by a slight change in the information of another agent j, and moreover, the preferences of i and j regarding (a, b) do not agree entirely in any arbitrary neighborhood around θ. Thus, locally around θ, there is preference interdependence because the preference of i depends non-trivially on j's information, and there is preference heterogeneity because the preferences of i and j differ.

There are three reasons why we suggest that the sufficient condition requires only a "small amount" of preference interdependence and heterogeneity. First, the condition only imposes restrictions on those "indifference" states where agents are actually indifferent between alternatives. Second, the "magnitude" of preference interdependence and heterogeneity, locally at a state, need not be large. Indeed, the condition is satisfied at θ even if i's preference is barely sensitive to j's information, and their preferences are almost but not entirely identical. Third, for an indifference state and a corresponding pair of alternatives, we merely need two agents, i and j, whose preferences are interdependent and heterogeneous. In other words, two agents are enough to disrupt ex post implementation.

The range of environments where our impossibility theorem applies is not only broad in theory, but also relevant in practice: decision-relevant information is often dispersed across individuals with diverse intentions and tastes, which formally translates into interdependence and heterogeneity of preferences. In terms of how mechanisms operate in the real world, our result therefore predicts that equilibrium outcomes are likely sensitive to what agents believe about each others' information.

Nonetheless, two prominent types of environments violate our sufficient condition: environments with *private values*, where agents' preferences never depend on the information of others, and environments with *common values*, where agents share the same preferences in every state. It is therefore not surprising that these environments admit non-constant EPIC social choice functions. In the case of private values, EPIC is known to be equivalent to strategy-proofness. There, dictatorships are strategy-proof, and more non-constant social choice functions become strategy-proof when the famous Gibbard-Satterthwaite Theorem

(Gibbard, 1973; Satterthwaite, 1975) is circumvented through restrictions on the underlying preference domain.[2] In the case of common values, the social choice function that chooses the common first-best alternative in each state is clearly EPIC. Yet, as we have seen, the possibility of ex post implementation quickly fades as we move away from these two extremes, when both preference interdependence and heterogeneity come into play. In particular, not even dictatorship is EPIC when values are interdependent,[3] and various exceptions to Gibbard-Satterthwaite are killed by even a small amount of preference interdependence.

There are only a few other papers on ex post implementation without transfers. Che, Kim, and Kojima (2015) and Fujinaka and Miyakawa (2020) as well as Pourpouneh, Ramezanian, and Sen (2020) study specific settings, namely object assignment and matching problems, respectively. We discuss how these specific settings relate to our result in Section IV. For more general settings, Barberà, Berga, and Moreno (2018, 2019) and Feng and Wu (2020, Section 4.3) also discuss necessary and sufficient conditions for the impossibility of ex post implementation. Unlike us, these papers impose no topological structure on the state space, and therefore their conditions are more abstract and harder to interpret and verify than ours.

This paper is organized as follows. Section I illustrates the main insight in a simple example. Section II sets up the general model. Section III presents the main result. Section IV discusses ex post implementation in situations where our result is silent : (1) allowing transfers; (2) matching and assignment problems; (3) discrete state spaces; (4) stochastic social choice with three or more alternatives. All proofs are in the Appendix.

I. Example

Two agents, 1 and 2, need to make a collective choice from two alternatives, S(afe) and R(isky), e.g., whether or not to pass a law, implement a project, or convict a defendant. The value of S is always 0 to both agents, whereas the value of R depends on an unknown state $\theta = (\theta_1, \theta_2)$, which can take any value from $\Theta = [-1, 1]^2$. Specifically, the value of R to agent $i = 1, 2$ is given by

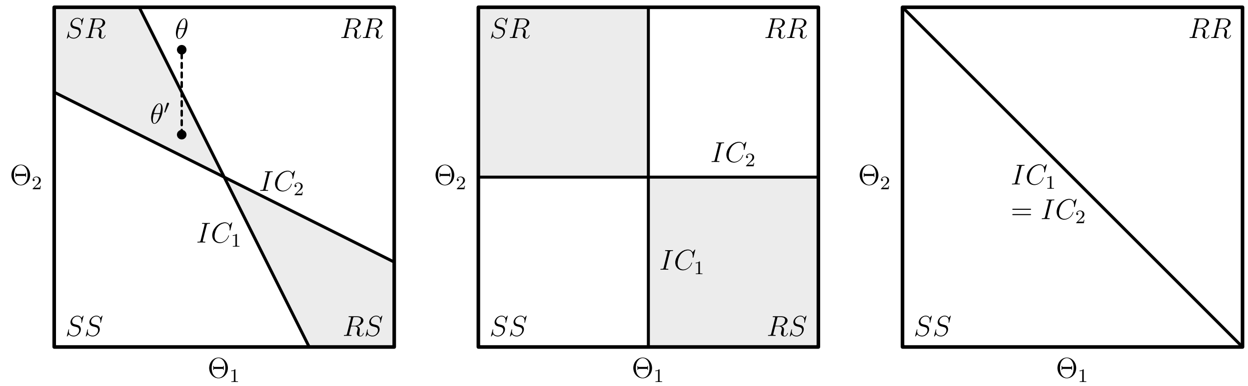
$$v_i^R(\theta) = \theta_i + \beta\theta_{-i}$$

where $\beta \in [0, 1]$.

Agent i can observe θ_i but not θ_{-i}. Thus, each agent only has partial information about the true payoff-relevant state, and β is a parameter that captures the degree to which agent

[2] See, for example, Moulin (1980) and Saporiti (2009).

[3] The reason is that a dictator who decides based on her own information would revise her choice in some states after learning about other agents' information. Also see Jehiel et al. (2006) for disambiguation of the term *dictatorship* in interdependent value environments.



(a) Interdependent Values (b) Private Values (c) Common Values

Figure 1: An illustration of the example

i's valuation depends on the information of the other agent $-i$. Note that when $\beta = 0$, this is a *private value* environment where an agent's preference depends only on her own information, whereas when $\beta = 1$, this is a *common value* environment where the agents preferences are identical. We will return to these special cases in a moment.

We first focus on an intermediate case $\beta = \frac{1}{2}$. Since each agent's valuation for R is twice as sensitive to her own information as to the other agent's information, the two agents do not always agree on which alternative is better. Indeed, in Figure 1a, which graphically represents the setting, the two agents' *indifference curves* $IC_i = \{\theta \mid v_i^R(\theta) = 0\}$, i.e., the respective sets of states where 1 and 2 are indifferent between S and R, partition the state space into four regions, $\{RR, RS, SR, SS\}$, where region XY has the interpretation that within it, agent 1 strictly prefers alternative X and agent 2 strictly prefers alternative Y.

Which deterministic social choice functions $\phi : [-1, 1]^2 \to \{S, R\}$ are EPIC when $\beta = \frac{1}{2}$? ϕ is EPIC if and only if it is optimal for each agent i to truthfully report her type θ_i to the direct mechanism induced by ϕ *in every state*, given that the other agent also reports truthfully. Obviously, any constant ϕ is EPIC. It turns out that the converse is also true: Any EPIC ϕ must be constant.

Let us briefly sketch the gist of the formal argument. Note that if an agent has the same preference across two states that differ only in her own information, then an EPIC social choice function must choose the same alternative in both states. As an example, consider the two states θ and θ' in Figure 1a. These states are aligned vertically (thus differ only in agent 2's information) and are respectively located in RR and SR (thus agent 2 strictly prefers R in both states). If some ϕ chose different alternatives in θ and θ', then agent 2 would be decisive in either state: she could induce the choice of one alternative by reporting her information truthfully, or the choice of the other alternative by misreporting her information to be dimension 2 of the other state. But since she strictly prefers R in both states, she would induce the choice of R in one of the states by misreporting her private information, contradicting EPIC.

Now, any EPIC ϕ must be constant within each of the four regions where both agents' preferences are strict and constant because we could otherwise find two states in the same region that differ only in one agent's information but where different alternatives are chosen, contradicting our previous observation about EPIC.

In addition, ϕ must choose the same alternative across any two adjacent regions because we can always find states such as θ and θ' that link two regions through an agent whose preference is the same. It follows that any EPIC ϕ must choose the same alternative across all four regions.[4]

It is worth noting that the linking argument across regions relies on the existence of the two states (θ, θ') that (1) differ only in one dimension $j \in \{1, 2\}$, and in which (2) agent $i \neq j$ has different ordinal preferences but (3) agent j has the same ordinal preference. Conditions (1) and (2) jointly entail *preference interdependence* between the agents: the change of agent j's information leads to a change in agent i's ordinal preference. Conditions (2) and (3) jointly entail *preference heterogeneity*: the agents' ordinal preferences do not always agree, so that a change in the state may cause a change in one agent's preference but not in the other's. In short, that ϕ is constant relies on the presence of preference interdependence and heterogeneity.

Not surprisingly, there exist non-constant EPIC ϕ if preference interdependence is absent as in the private value case $\beta = 0$ (Figure 1b) or if preference heterogeneity is absent as in the common value case $\beta = 1$ (Figure 1c) because we cannot find the desired linking states (θ, θ') in either case. For example, the function ϕ that chooses R only in RR is EPIC in both cases.

On the other hand, the argument goes through for any $\beta \in (0, 1)$, i.e., when there is at least some preference interdependence and heterogeneity, regardless of how close β is to one of the two exceptional cases. In this sense, if we think of the environments with interdependent values as a spectrum parametrized by $\beta \in [0, 1]$ with private values at one end and common values at the other, then even a slight departure from the two extremes leads to an impossibility of ex post implementation. This insight, as formalized and generalized in Theorem 1, is the main contribution of the paper.

[4] In this example, it is easy to show that ϕ must then also choose the same alternative on the indifference curves IC_1 and IC_2. In general, one can only show this for the interior of the state space; see the Appendix.

II. Model

A group of agents $N = \{1, \ldots, n\}$ must collectively choose an alternative from a finite set A without using monetary transfers. The valuation of agent $i \in N$ for alternative $a \in A$ depends on an underlying *state* $\theta \in \Theta$, where Θ is the set of all possible states. We represent i's valuation for alternative a by a *valuation function* $v_i^a : \Theta \to \mathbb{R}$. In addition, we let $v_i^{ab}(\theta) := v_i^a(\theta) - v_i^b(\theta)$ denote i's *relative valuation function* for a versus another alternative b. Thus, i weakly prefers a over b in state θ if and only if $v_i^{ab}(\theta)$ is non-negative.

Preference interdependence among the agents is typically modeled by assuming that each agent is only partially informed about the payoff-relevant state θ. Specifically, θ consists of n components, $\theta = (\theta_1, ..., \theta_n)$, and each agent i only observes θ_i — her *type.* The state space Θ is therefore $\prod_{i \in N} \Theta_i$. We assume $\Theta_i = [-1, 1]^{d_i}$ where $d_i \in \mathbb{N}$ is the dimension of agent i's type and thus allow for multidimensional types.[5]

Valuation functions are continuously differentiable. Given a relative valuation function v_i^{ab}, let ∇v_i^{ab} denote its gradient, and let $\nabla_{\theta_j} v_i^{ab}$ denote the d_j-dimensional vector of components of ∇v_i^{ab} with respect to the type of agent j. We follow Jehiel et al. (2006) in assuming that an agent's indifference between two alternatives is broken by a slight change in her own information. More precisely,

$$\forall i \in N, \forall \theta \in \Theta, \forall a, b \in A : \; a \neq b, \quad \left(v_i^{ab}(\theta) = 0 \implies \nabla_{\theta_i} v_i^{ab}(\theta) \neq \mathbf{0}\right).^6 \tag{RESP}$$

As motivated in the introduction, we are interested in the ex post implementability of social choice functions. By the Revelation Principle, we can focus on those that are truthfully ex post implementable in direct mechanisms, or in other words, ex post incentive compatible. Specifically, a (deterministic) *social choice function* $\phi : \Theta \to A$ is *ex post incentive compatible (EPIC)* if truth-telling is an ex post equilibrium of the direct mechanism induced by ϕ, i.e.

$$\forall i \in N, \forall \theta \in \Theta, \forall \tilde{\theta}_i \in \Theta_i, \quad v_i^{\phi(\theta_i, \theta_{-i})}(\theta) \geq v_i^{\phi(\tilde{\theta}_i, \theta_{-i})}(\theta). \tag{EPIC}$$

Following Jehiel et al. (2006), we say that social choice function ϕ is *trivial* if it is constant on the interior of Θ.

[5] Our result still obtains if each Θ_i is a subset of a Euclidean space with connected interior. Moreover, Θ need not be a product state space, provided its interior is connected.

[6] This assumption is not necessary for the gist of our result but simplifies statement and proof: without (RESP), the result's conclusion must be slightly weakened, making the result harder to communicate. See Feng and Wu (2020) for an earlier version of the result without (RESP).

III. Impossibility of ex post implementation

Let us first formally present the main result and explain it in more detail right after. For a pair of distinct alternatives (a, b), let

$$I^{ab}(\theta) = \{i \in N \mid v_i^{ab}(\theta) = 0\}$$

denote the set of agents who are indifferent between this pair in state θ. If $I^{ab}(\theta)$ is nonempty, we say that (a, b) is an *indifference pair* of θ. Moreover, we say that θ is an *indifference state* if it has at least one indifference pair.

Theorem 1. *Suppose for any indifference state θ and any of its indifference pairs (a, b), there exists an agent $i \in I^{ab}(\theta)$ and another agent $j \in N$ such that:*
(1) (local interdependence) $\nabla_{\theta_j} v_i^{ab}(\theta) \neq \mathbf{0}$;
(2) (local heterogeneity) $j \notin I^{ab}(\theta)$ *or* $\nabla v_i^{ab}(\theta) \neq \lambda \nabla v_j^{ab}(\theta)$ *for any* $\lambda \geq 0$.
Then, all EPIC social choice functions are trivial.

To better understand the result, let us parse the statement. Note first that the sufficient condition only constrains indifference states regarding their indifference pairs. That is, only for the indifference states θ and their indifference pairs (a, b) do we need to find two agents i and j whose preferences regarding (a, b) are interdependent but nonetheless heterogeneous locally around θ. More precisely, local interdependence means that i, who is indifferent between (a, b) in θ, is no longer indifferent following some small change in j's type, i.e., the ordinal preference of i depends on j's information around θ. Local interdependence is satisfied in Figure 1a but not in Figure 1b because it requires agent 1's indifference curve IC_1 to not be entirely vertical and agent 2's indifference curve IC_2 to not be entirely horizontal. Local heterogeneity means that i and j disagree on whether a or b is better in or near state θ. Specifically, if $j \notin I^{ab}(\theta)$, then heterogeneity in θ is immediate: i is indifferent, but j is not. On the other hand, if j is also indifferent in θ, then the condition that $\nabla v_i^{ab}(\theta) \neq \lambda \nabla v_j^{ab}(\theta)$ for any $\lambda \geq 0$, i.e., that the two gradients are not co-directional at θ, implies that there is an arbitrarily close state in which i and j rank (a, b) differently.[7] Local heterogeneity is satisfied in Figure 1a but not in Figure 1c because it requires that IC_1 and IC_2 cross each other when they intersect,[8] as only then would the gradients, which are respectively normal to the indifference curves, be misaligned at the intersection.

[7] Another way to think of this condition is that there are two pieces of information in state θ between which i and j have different marginal rates of substitution.

[8] For local heterogeneity to hold at the intersection, the indifference curves of i and j may be tangent only when their preferences regarding (a, b) are diametrically opposed in a neighborhood of the intersection, which is not possible in the example for any $\beta \in [0, 1]$.

The strength of Theorem 1 lies in the weakness of its sufficient condition. First, the condition only puts restrictions on indifference states.[9] Second, local interdependence only rules out the knife-edge case that $\nabla_{\theta_j} v_i^{ab}(\theta)$ is exactly equal to $\mathbf{0}$, and likewise, in case $j \in I^{ab}(\theta)$, local heterogeneity only rules out the knife-edge case that $\nabla v_i^{ab}(\theta)$ and $\nabla v_j^{ab}(\theta)$ are exactly co-directional. In other words, the sufficient condition is satisfied even if, locally around θ, there is only a minimal amount of preference interdependence and heterogeneity. Third, for there to be local interdependence and heterogeneity, we only need two agents whose preferences jointly satisfy the respective requirements, and these agents need not be the same across indifference states or even pairs.[10]

In fact, the result can be further strengthened. First, what we prove in the Appendix is actually stronger (Theorem 2): non-trivial social choice functions do not exist even under the weaker notion of *local ex post incentive compatibility*, which requires that no agent i has an incentive to slightly misrepresent her true type θ_i as some $\tilde{\theta}_i$ that is close to θ_i. Moreover, the presence of local interdependence and heterogeneity in *every* indifference state is an overkill for deriving the impossibility result. All that is needed is a finite number of indifference states satisfying the conditions; see Remark 1 in the Appendix.

IV. Discussion

Finally, we discuss some limits of our impossibility theorem and show that they correspond to situations where non-trivial ex post implementation has a better chance.

A. Transfers

In the Introduction, we mentioned that transfers facilitate ex implementation. When transfers are allowed and an agent only cares about her own transfer, as is typically assumed, she is indifferent between any two outcomes where the chosen non-monetary alternative and her own transfer are the same, despite differences in the other agents' transfers. These indifferences persist across states and thus violate both local interdependence and (RESP), rendering our result silent. Transfers can be used to overcome preference interdependence or heterogeneity— the two roadblocks suggested by our result— by either making values effectively private or by aligning the agents' interests.[11] For instance, if utility is quasi-linear in our leading example, then making each agent i pay $\beta\theta_{-i}$ removes interdependence, while paying each agent $(1-\beta)\theta_{-i}$ removes heterogeneity.

[9] Clearly, for any continuous distribution on Θ, the set of indifference states has measure zero.

[10] In particular, our result still holds if subsets of agents, say, parties in a parliament, have identical preferences, as long as there is preference interdependence and heterogeneity between parties.

[11] See section 5 in Chung and Ely (2003) for further discussion on how transfers can be used to align individual interests in collective choice problems.

B. Assignment and Matching Problems

A common assumption in matching is that each agent only cares about her own assigned object or match. Thus, local interdependence and (RESP) generally fail to hold in such problems.[12] It is therefore not surprising that non-trivial EPIC social choice functions exist even when preferences are interdependent.[13] However, as Che, Kim, and Kojima (2015) show, such EPIC social choice functions cannot be efficient, at least in the housing allocation problem where each agent is assigned exactly one object. Moreover, our negative result can still apply to assignment or matching problems with allocative externalities, e.g., when students not only care about which dorm room they get but also which rooms their friends get.

C. Discrete State Spaces

We have assumed that the state space is a connected subset of a Euclidean space. If instead the state space is discrete, then counterexamples to our result are easy to find. For instance, see Feng and Wu (2020). One way to understand the discrepancy between discrete and continuous state spaces is to think of a discrete state space as a low-resolution discretization of a continuous space. For example, suppose each agent's actual type can be any number between -1 and 1, yet each agent is only aware of whether her type is above or below 0, making her effective type space binary. Since the agents' indifference curves are then being squeezed into a discrete grid, they tend to become more aligned, and this alignment gives leeway to non-trivial ex post implementation.

D. Stochastic Social Choice Functions

What if we allow for randomization so that the collective choice can be a *lottery* over alternatives? It turns out that Theorem 1 still holds as long as there are only two alternatives. The reason is simple: an agent is indifferent between lotteries if and only if she is indifferent between the two underlying alternatives, and she otherwise prefers lotteries in which her preferred alternative is chosen with a higher rather than lower probability. Thus, our arguments immediately extend to stochastic implementation with two alternatives. However, if there are three or more alternatives, then an agent can get the same expected utility from different lotteries despite having a strict preference over the underlying alternatives. In the following example, these indifferences can indeed be used to construct a non-trivial stochastic EPIC

[12] The housing allocation problem with two objects and two agents is an exception since the assignment of one object to one agent implies that the remaining object must be assigned to the other agent. See also the illustrative example in Che, Kim, and Kojima (2015).

[13] This observation echoes how Jehiel et al. (2006) relies on allocative externalities; also see Bikhchandani (2006).

social choice function.

Example (continued)**.** Agents 1 and 2 now decide between three alternatives, R, S, and P. For $i = 1, 2$, still assume $\Theta_i = [-1, 1]$, $v_i^R = \theta_i + \frac{1}{2}\theta_{-i}$, and $v_i^S = 0$. Additionally, assume $v_i^P = -1$. Theorem 1 applies here, so non-trivial deterministic EPIC social choice functions do not exist. However, consider the stochastic social choice function $\phi = (\phi^R, \phi^P, \phi^S)$ given by

$$\phi^R(\theta) = \frac{4 + 2\theta_1 + 2\theta_2}{11}, \quad \phi^P(\theta) = \frac{\theta_1^2 + \theta_1\theta_2 + \theta_2^2}{11}, \quad \phi^S(\theta) = 1 - \phi^S(\theta) - \phi^P(\theta)$$

where $\phi^X(\theta)$ denotes the probability that alternative X will be chosen in state θ. It is readily verified that ϕ is EPIC.

Appendix: Proof of Theorem 1

Endow Θ with the norm topology. Let $B_\varepsilon(\theta)$ denote the open ball with radius $\varepsilon > 0$ centered at θ. A social choice function ϕ is said to be *locally EPIC* if there exists some $\varepsilon > 0$ such that for any $\theta \in \Theta$, ϕ restricted to $B_\varepsilon(\theta)$ is EPIC, i.e.

$$\forall i \in N, \forall \theta \in \Theta, \forall \tilde{\theta}_i \in \Theta_i : (\tilde{\theta}_i, \theta_{-i}) \in B_\varepsilon(\theta), \quad v_i^{\phi(\theta_i,\theta_{-i})}(\theta) \geq v_i^{\phi(\tilde{\theta}_i,\theta_{-i})}(\theta). \qquad \text{(LEPIC)}$$

Let $\overline{\Theta} := \{\theta \in \operatorname{int}\Theta \mid \forall i \in N, \forall a, b \in A : a \neq b,\ v_i^{ab}(\theta) \neq 0\}$ denote the set of interior states where all agents have strict preferences. $\overline{\Theta}$ is open because valuation functions are continuous. Similarly, each connected component of $\overline{\Theta}$ is open. Note that the ordinal preferences of all agents are strict and constant on each connected component of $\overline{\Theta}$.

Lemma 1. *If ϕ is locally EPIC, then ϕ is constant on each connected component of $\overline{\Theta}$.*

Proof. Suppose ϕ satisfies (LEPIC) for some $\varepsilon > 0$. Let C be a connected component of $\overline{\Theta}$. Suppose for the sake of contradiction that ϕ is not constant on C, then there exists some $\theta \in C$ and $\tilde{\varepsilon} \in (0, \varepsilon)$ such that $B_{\tilde{\varepsilon}}(\theta) \subset C$ and $\phi(\theta) \neq \phi(\theta')$ for some $\theta' \in B_{\tilde{\varepsilon}}(\theta)$. Clearly we can find a sequence $(\theta^0, ..., \theta^n)$ in $B_{\tilde{\varepsilon}}(\theta)$ where $\theta^0 = \theta$, $\theta^n = \theta'$, and for every $k = 0, ..., n-1$, θ^k and θ^{k+1} differ at most in the $k+1$th entry. Thus $\phi(\theta) \neq \phi(\theta')$ implies that $\phi(\theta^k) \neq \phi(\theta^{k+1})$ for some k. By construction, θ^k and θ^{k+1} differ only in the type of agent $k+1$ who has the same strict ordinal preferences in both states. Therefore, she either could profit from misreporting her type as θ_{k+1}^k in state θ^{k+1} or from misreporting her type as θ_{k+1}^{k+1} in state θ^k, contradicting (LEPIC). □

Given Lemma 1, it causes no confusion to denote $\phi(C)$ as the choice by ϕ on a connected component C of $\overline{\Theta}$.

Two connected components C, C' of $\overline{\Theta}$ are said to be *adjacent at* $\theta \in \operatorname{int}\Theta$ if

1. $\theta \in \operatorname{cl} C \cap \operatorname{cl} C'$, and
2. $B_\varepsilon(\theta) \subset \operatorname{cl} C \cup \operatorname{cl} C'$ for some $\varepsilon > 0$.

A collection $\mathbf{X}$ of vectors are said to be *collinear* if for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} = \lambda \mathbf{y}$ for some $\lambda \in \mathbb{R}$, i.e., these vectors lie on a common line passing through the origin. If, in addition, for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} = \lambda \mathbf{y}$ for some $\lambda \geq 0$, i.e., these vectors lie on a common ray emanating from the origin, then they are said to be *co-directional.*

Lemma 2. *Suppose ϕ is locally EPIC. If C, C' are two connected components of $\overline{\Theta}$ adjacent at $\theta \in \operatorname{int}\Theta$, and $\phi(C) := a \neq b =: \phi(C')$, then*
(1) $\nabla_{\theta_j} v_i^{ab}(\theta) = \mathbf{0}$ for any $i \in I^{ab}(\theta)$ and $j \in N \setminus I^{ab}(\theta)$, and
(2) $\left(\nabla v_i^{ab}(\theta)\right)_{i \in I^{ab}(\theta)}$ are co-directional.

Proof. The lemma's premises imply that we can find $\varepsilon > 0$ such that (1) (LEPIC) holds for $B_\varepsilon(\theta)$, (2) $B_\varepsilon(\theta) \subset \operatorname{cl} C \cup \operatorname{cl} C'$, and (3) for any agent i and any distinct pair of alternatives (x, y), if i strictly prefers x to y in θ, then she strictly prefers x to y in every state in $B_\varepsilon(\theta)$.

Arbitrarily pick alternatives $x, y, w, z \in A$ where $x \neq y$ and $w \neq z$ and agents $i \in I^{xy}(\theta)$ and $j \in I^{wz}(\theta)$. Claim that $\nabla v_i^{xy}(\theta)$ and $\nabla v_j^{wz}(\theta)$ are collinear. Indeed, if not, then we can find $\theta', \theta'' \in B_\varepsilon(\theta)$ such that $v_i^{xy}(\theta') = 0$ but $v_j^{wz}(\theta') \neq 0$, and $v_i^{xy}(\theta'') \neq 0$ but $v_j^{wz}(\theta'') = 0$.

By (RESP), we can find two states arbitrarily close to θ' (hence within $B_\varepsilon(\theta)$) in which j has the same strict preference regarding (w, z) but i has different strict preferences regarding (x, y). Similarly, we can find two states arbitrarily close to θ'' (hence also within $B_\varepsilon(\theta)$) in which i has the same strict preference regarding (x, y) but j has different strict preferences regarding (w, z). Thus $B_\varepsilon(\theta)$ must intersect at least three distinct connected components of $\overline{\Theta}$ as it contains at least three profiles of strict preferences of the agents. This contradicts that $B_\varepsilon(\theta)$ only intersects two such components, namely C and C'.

Towards proving part (1), suppose for the sake of contradiction that there exists $i \in I^{ab}(\theta)$ and $j \in N \setminus I^{ab}(\theta)$ such that $\nabla_{\theta_j} v_i^{ab} \neq \mathbf{0}$. Thus, we can find $\rho > 0$ sufficiently small such that $\theta' := (\theta_j + \rho \nabla_{\theta_j} v_i^{ab}(\theta),\ \theta_{-j}) \in B_\varepsilon(\theta)$, $\theta'' := (\theta_j - \rho \nabla_{\theta_j} v_i^{ab}(\theta),\ \theta_{-j}) \in B_\varepsilon(\theta)$, and

$$v_i^{ab}(\theta') v_i^{ab}(\theta'') < 0, \quad v_j^{ab}(\theta') v_j^{ab}(\theta'') > 0.$$

In other words, agent i has different strict preferences regarding (a, b) in θ' and θ'', whereas agent j has the same strict preference. By the collinearity observation above, we can further conclude that, for ρ small enough, any agent k who is indifferent between any pair (x, y) in θ has different strict preferences regarding this pair in θ' and θ''. Together with $\theta', \theta'' \in B_\varepsilon(\theta)$ we thus establish $\theta', \theta'' \in \overline{\Theta}$, i.e., all agents have strict preferences in both states. Moreover, θ' and θ'' must be in distinct connected components of $\overline{\Theta}$ — one in C, the other in C' — because i's preferences differ across the two states. Since agent j has the same strict preference regarding (a, b) in θ' and θ'' and since the two states differ only in j's type, (LEPIC) implies $\phi(\theta') = \phi(\theta'')$, a contradiction.

Now we show part (2). From the collinearity observation we conclude that $\nabla v_i^{ab}(\theta)$ and $\nabla v_j^{ab}(\theta)$ are collinear for any $i, j \in I^{ab}(\theta)$. If, for the sake of contradiction, for some $i, j \in I^{ab}(\theta)$ the two gradients are not also co-directional, then they must be diametrically opposed. By (RESP), we can find $\rho > 0$ sufficiently close to 0 such that the following three statements are true. First, i strictly prefers a to b and j strictly prefers b to a in both of the following two states:

$$\hat{\theta} := (\theta_i + \rho \nabla_{\theta_i} v_i^{ab}(\theta),\ \theta_{-i}) \quad \text{and} \quad \tilde{\theta} := (\theta_j - \rho \nabla_{\theta_j} v_j^{ab}(\theta),\ \theta_{-j}).$$

Second, i strictly prefers b to a and j strictly prefers a to b in both of the following two states:

$$\hat{\theta}' := (\theta_i - \rho \nabla_{\theta_i} v_i^{ab}(\theta),\ \theta_{-i}) \quad \text{and} \quad \tilde{\theta}' := (\theta_j + \rho \nabla_{\theta_j} v_j^{ab}(\theta),\ \theta_{-j}).$$

Third, the above two pairs of states are in $B_\varepsilon(\theta)$. Following the argument in the previous paragraph, the four states are also in $\overline{\Theta}$ and thus either in C or in C'. In addition, one pair must fall in C and the other pair must fall in C' because the preferences of agent i (equivalently, j) regarding (a, b) are the same within each pair but differ across pairs. Therefore, $\phi(\hat{\theta}) = \phi(\tilde{\theta})$ but $\phi(\hat{\theta}) \neq \phi(\hat{\theta}')$. (LEPIC) implies that $\phi(\hat{\theta}) = a$ for otherwise i would misreport her type as $\hat{\theta}_i'$ in state $\hat{\theta}$. Similarly, $\phi(\tilde{\theta}) = b$ for j not to misreport, but then $\phi(\hat{\theta}) \neq \phi(\tilde{\theta})$, a contradiction. □

Lemma 3. *For any two connected components C and C' of $\overline{\Theta}$ there exists a finite sequence of connected components $C^0, \ldots, C^K$ of $\overline{\Theta}$ and a finite sequence of indifference states $\theta^1, \ldots, \theta^K \in \operatorname{int}\Theta$ such that $C^0 = C, C^K = C'$, and C^k and C^{k+1} are adjacent at θ^{k+1} for every $k = 0, \ldots, K-1$.*

Proof. Since $\operatorname{int}\Theta$ is connected, by a standard local-to-global argument it suffices to show that for any $\theta \in \operatorname{int}\Theta$ there exists $\varepsilon > 0$ such that the lemma holds on $B_\varepsilon(\theta)$, which, if θ is not an indifference state, is immediate. To show the same for an indifference state, first note that for any sufficiently small open ball $B \subset \operatorname{int}\Theta$, if B intersects an *indifference set* $IC_i^{ab} := \{\tilde{\theta} \in \Theta \mid v_i^{ab}(\tilde{\theta}) = 0\}$ for some $i \in N$ and $a, b \in A$, $a \neq b$, then (RESP) and the implicit function theorem jointly imply that $B \cap IC_i^{ab}$ is a connected hypersurface within B. Moreover, IC_i^{ab} *separates* B into two open connected components U and U', i.e., $B \setminus IC_i^{ab} = U \cup U'$ and $IC_i^{ab} \cap B \subset \operatorname{cl} U \cap \operatorname{cl} U'$.

Pick any indifference state θ and $\varepsilon > 0$ sufficiently small. Let $\mathcal{C}$ denote the set of connected components of $\overline{\Theta} \cap B_\varepsilon(\theta)$, which is finite by the previous paragraph and the fact that there are only finitely many agents and alternatives. Thus, the lemma holds on $B_\varepsilon(\theta)$ if any arbitrary bipartition $\{\mathcal{P}, \mathcal{P}'\}$ of $\mathcal{C}$ has $C \in \mathcal{P}$ and $C' \in \mathcal{P}'$ adjacent to each other, which can be established if we can find an open connected $U^* \subset B_\varepsilon(\theta)$ that intersects with exactly one $C \in \mathcal{P}$ and one $C' \in \mathcal{P}$.

We search for U^* iteratively. Let $U^0 = B_\varepsilon(\theta)$. If $\#\mathcal{C} = 2$, then $U^0 = U^*$ suffices. If $\#\mathcal{C} \geq 3$, then we can find some hypersurface IC_i^{ab} such that an open connected component U^1 of $U^0 \setminus IC_i^{ab}$ contains some $C \in \mathcal{P}$ and some $C' \in \mathcal{P}'$, for otherwise every hypersurface would identically separate $\mathcal{C}$ into $\{\mathcal{P}, \mathcal{P}'\}$, which would imply that some distinct $C, C' \in \mathcal{C}$ are not separated by any hypersurface, a contradiction. If U^1 contains just two connected components in $\mathcal{C}$, then $U^1 = U^*$ as desired; otherwise, construct U^2 from U^1 in the same way U^1 was constructed from U^0, and so forth.[14] Since $\mathcal{C}$ is finite, we eventually obtain the desired U^*. $\square$

We will now state and prove a stronger impossibility theorem which immediately implies Theorem 1 as a corollary.

Theorem 2. *Suppose the premises of Theorem 1 hold. Then, all* ***locally*** *EPIC social choice functions are trivial.*

Proof. Fix any ϕ that is locally EPIC for radius $\varepsilon > 0$. Let $\Theta^k \subset \operatorname{int}\Theta$ denote the set of interior states where exactly k agents have indifferences in their preferences. Thus $\operatorname{int}\Theta = \bigcup_{k=0}^{n} \Theta^k$. It suffices to show that ϕ is constant on Θ^k for every $k = 0, ..., n$ and, moreover, that $\phi(\Theta^0) = ... = \phi(\Theta^n)$. We proceed by induction on k.

For $k = 0$, note that $\Theta^k = \overline{\Theta}$. Suppose, for the sake of contradiction, that ϕ is not constant on $\overline{\Theta}$. By Lemma 3, there exist two connected components C and C' of $\overline{\Theta}$ adjacent at some indifference state θ such that $\phi(C) \neq \phi(C')$. For any indifference pair (a, b) of θ, one of the following two cases must hold by assumption: (1) There is $i \in I^{ab}(\theta)$ and $j \notin I^{ab}(\theta)$ such that $\nabla_{\theta_j} v_i^{ab}(\theta) \neq \mathbf{0}$. (2) There are $i, j \in I^{ab}(\theta)$ where $\nabla v_i^{ab}(\theta)$ and $\nabla v_j^{ab}(\theta)$ are not co-directional. Hence we have $\phi(C) = \phi(C')$ by the contrapositive of Lemma 2, a contradiction. Thus ϕ must be constant on $\overline{\Theta} = \Theta^0$.

Now suppose ϕ is constant on Θ^ℓ for every $\ell < k$ and, moreover, $\phi(\Theta^0) = ... = \phi(\Theta^{k-1})$. Pick any $\theta \in \Theta^k$. By iteratively using (RESP), we can find states $\theta', \theta'' \in B_\varepsilon(\theta)$ arbitrarily close to θ such that (1) θ, θ' and θ'' differ from each other only in some agent i's type, (2) agent i is indifferent between one or more pairs of distinct alternatives in θ, and, in addition, for any such pair she has strict and opposite preferences in θ' and θ'', (3) for any

[14] If another hypersurface IC_j^{xy} is tangent to IC_i^{ab} at θ, then IC_j^{xy} may be disconnected in U^1. In this case, treat each connected component of $IC_j^{xy} \cap U^1$ as a separate hypersurface.

agent whose preference regarding any given pair of distinct alternatives is strict in θ, her preference regarding this pair remains the same in θ' and θ''. Thus $\theta' \in \Theta^{\ell}$ and $\theta'' \in \Theta^{\ell'}$ for $\ell, \ell' < k$. Consequently, the inductive hypothesis implies that $\phi(\theta') = \phi(\theta'') = \phi(\Theta^0)$. Suppose for the sake of contradiction that $\phi(\theta) = a$ but $\phi(\Theta^0) = b \neq a$. On the one hand, if i has a strict preference regarding (a, b) in θ, then she has the same strict preference in θ and θ', and hence by (LEPIC), we must have $a = \phi(\theta) = \phi(\theta') = b$ for there to be no incentive for i to misreport, a contradiction. On the other hand, if i is indifferent between (a, b) in θ, then, by construction, i strictly prefers a over b in one of θ' and θ'', and in that state, she has an incentive to misreport her type as θ_i, also a contradiction. Thus $\phi(\theta) = \phi(\Theta^0)$. Since θ was arbitrarily chosen from Θ^k, we conclude that ϕ must be constant on Θ^k and, moreover, $\phi(\Theta^k) = \phi(\Theta^0)$. $\square$

Remark 1. The sufficient condition for Theorems 1 and 2 can be weakened. Indeed, if Θ is compact, then Lemma 3 implies the existence of a finite sequence $C^0, ..., C^K$ that includes *all* connected components of $\overline{\Theta}$ and an associated sequence of indifference states $\theta^1, ..., \theta^K$ such that C^k and C^{k+1} are adjacent at θ^{k+1} for all $k = 0, ..., K-1$. We may call such a sequence of indifference states a *traversing sequence*. Clearly, the proof of Theorem 2 goes through as long as local interdependence and heterogeneity are present in states along a traversing sequence. Therefore, the sufficient condition can be weakened as follows: There exists a traversing sequence of indifference states where for each of the states and each of its indifference pairs, local interdependence and heterogeneity are present.

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